Explicit two-sided unique-neighbor expanders

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Joint work with

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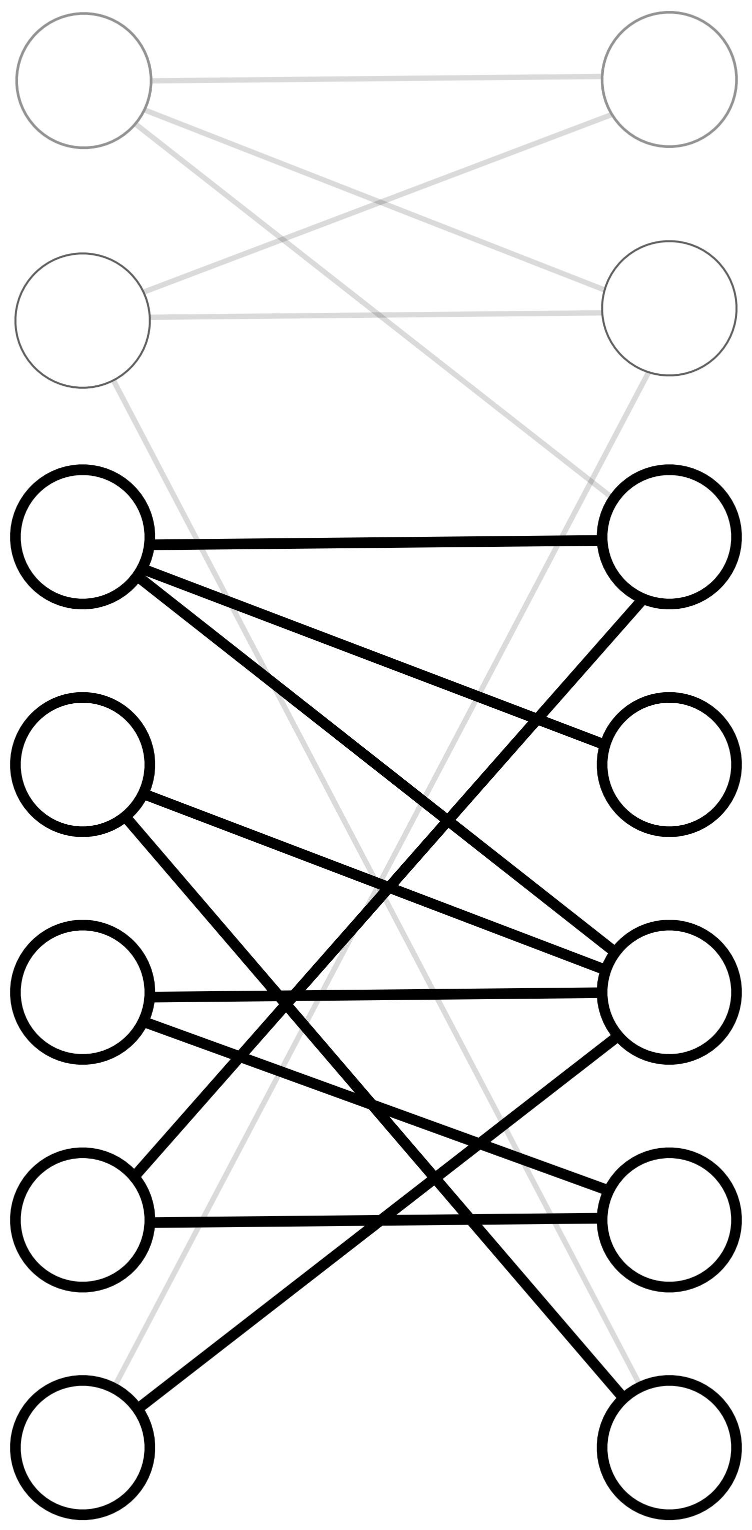
MIT

Theo McKenzie
Stanford

Pedro Paredes
Princeton

Unique-neighbor

G graph, $S \subseteq V(G)$



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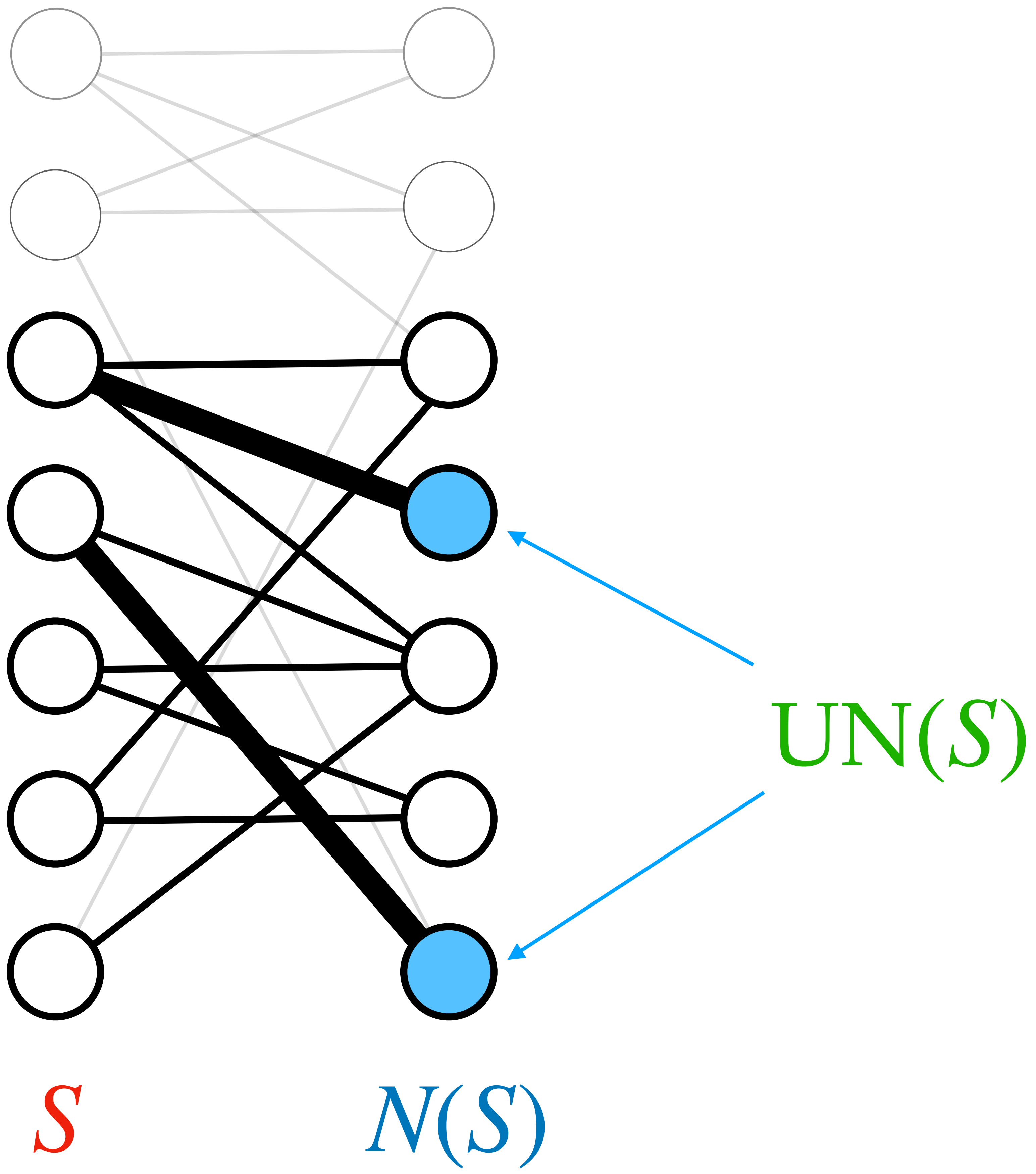
Unique-neighbor of S
vertex with exactly *one* edge to S.

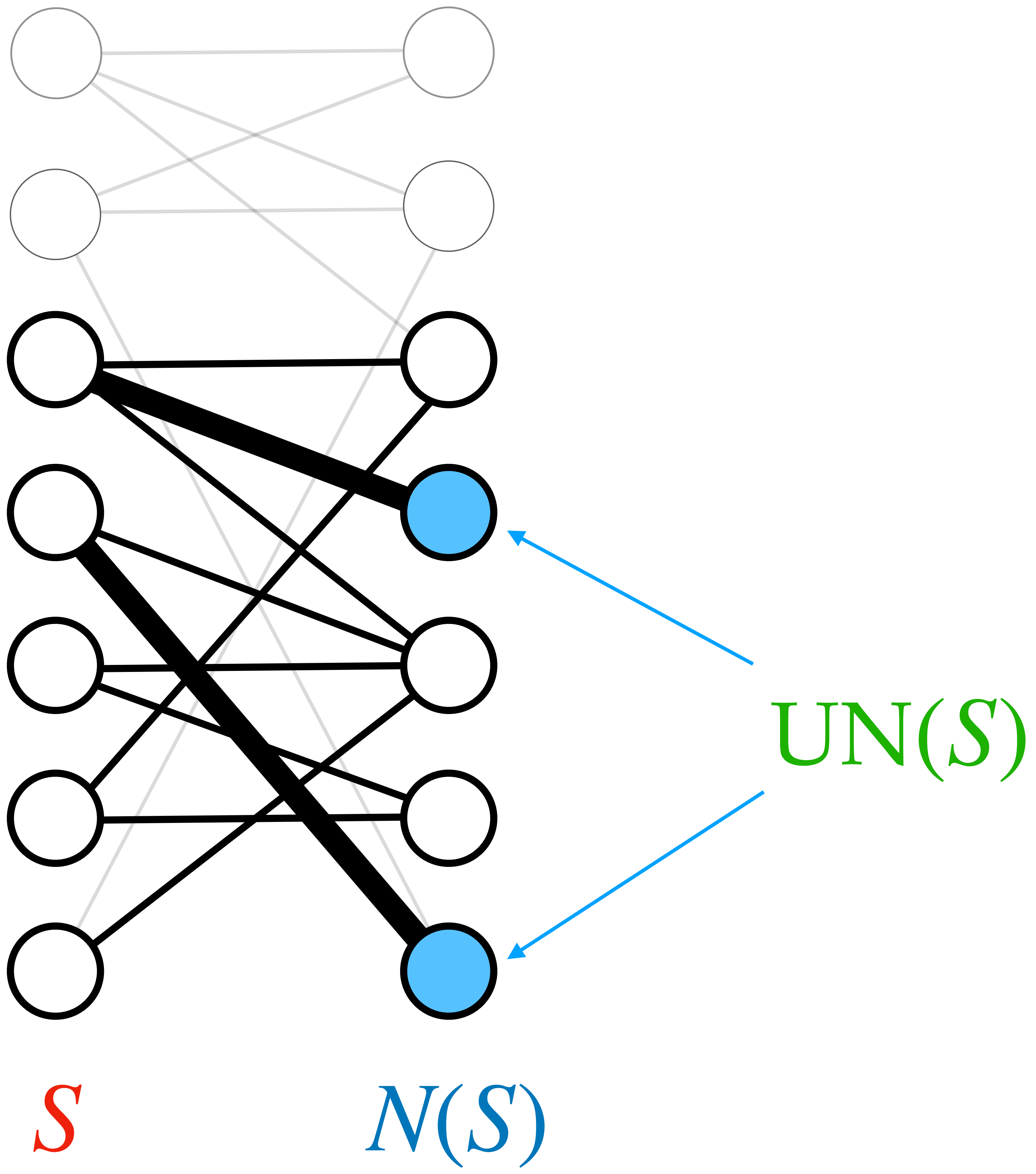


Unique-neighbor expander

G graph, $S \subseteq V(G)$

Unique-neighbor *of* S
vertex with exactly ***one*** edge to S.

Unique-neighbor expander
every small set has *many* unique-neighbors.



Unique-neighbor expander

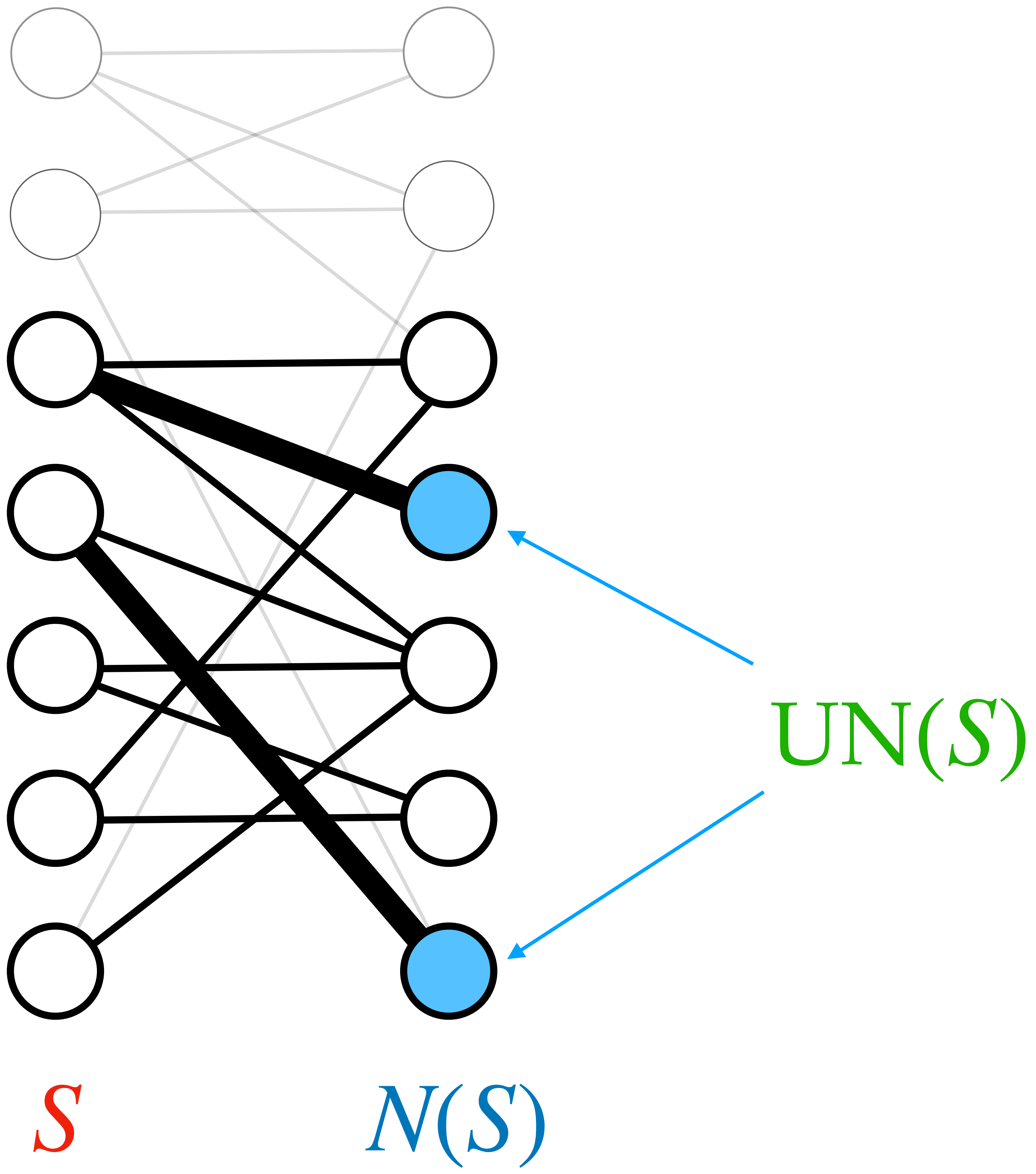
(1-sided) ***Unique-neighbor expander***

$\forall S \subseteq L$ s.t. $|S| \leq \delta |L|$:

$|\mathrm{UN}(S)| \geq \gamma \cdot d |S|$

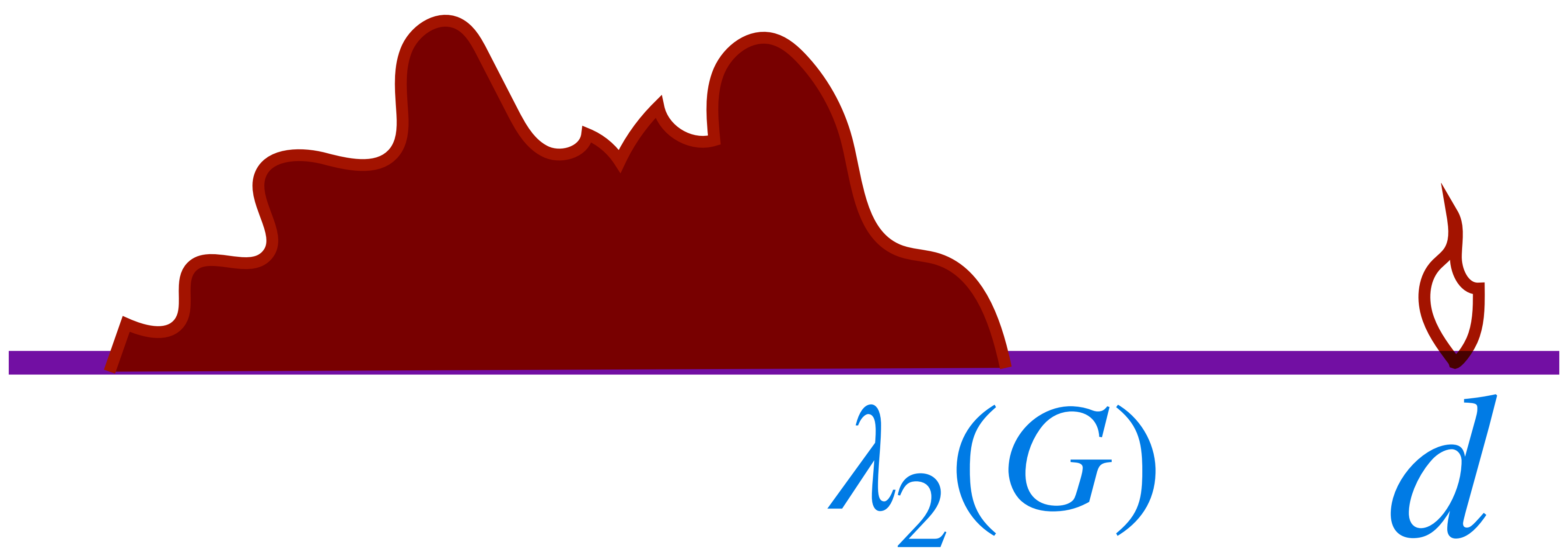
$\delta, \gamma > 0$ constants,

d degree.



Spectral expansion

Eigenvalues of adjacency matrix of G



$\lambda_2(G)$ is *spectral expansion* of G

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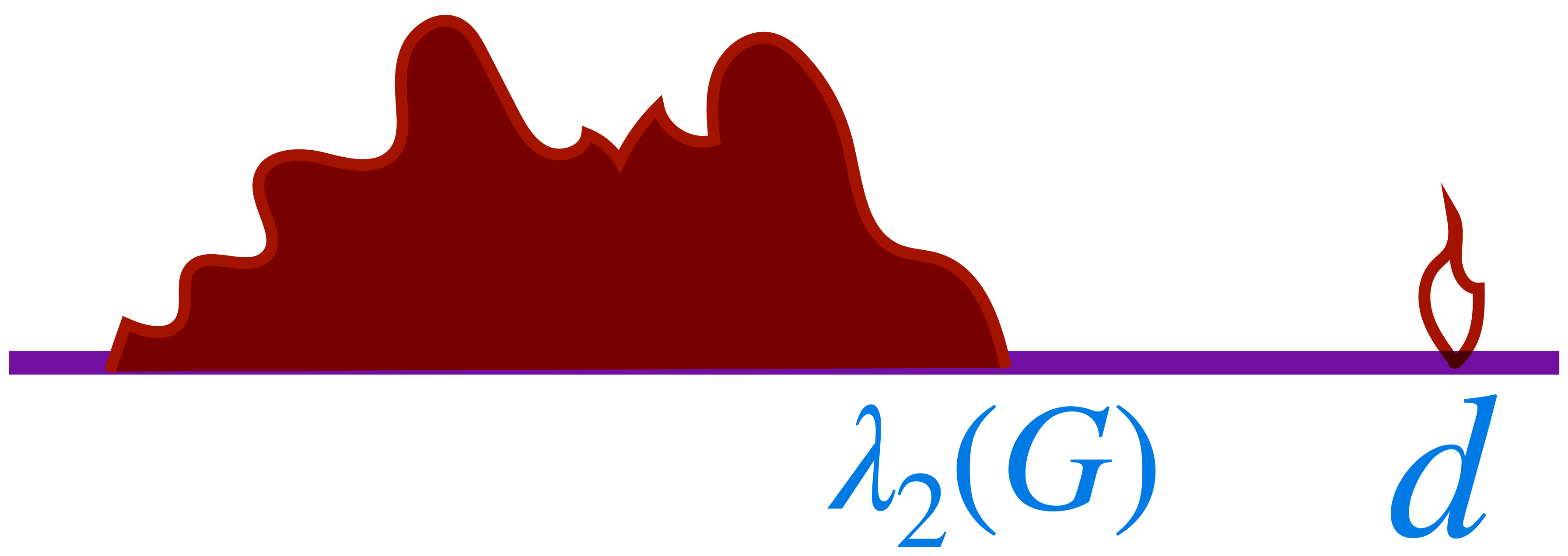

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Spectral expansion implies many *nice* combinatorial properties!

- *no sparse cuts*
- *no dense small subgraphs*
- *rapid mixing of random walks*

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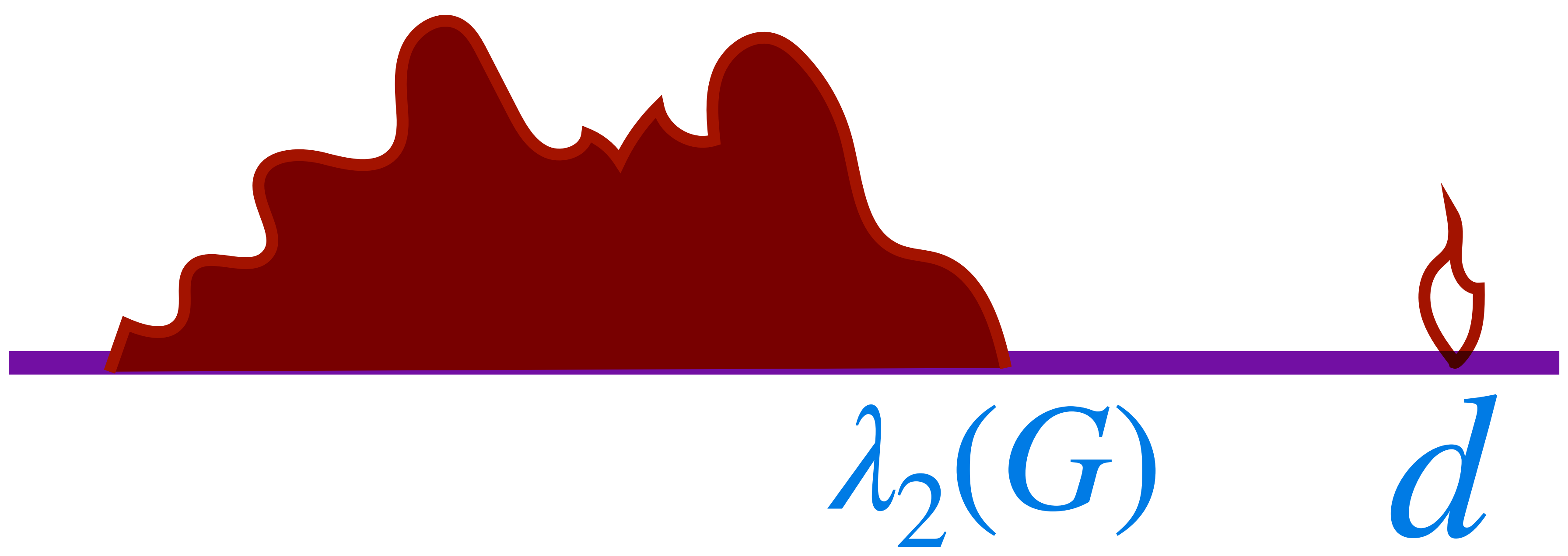

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Many known constructions!

Cayley graphs, zig-zag product, derandomization, interlacing polynomials

Spectral expansion

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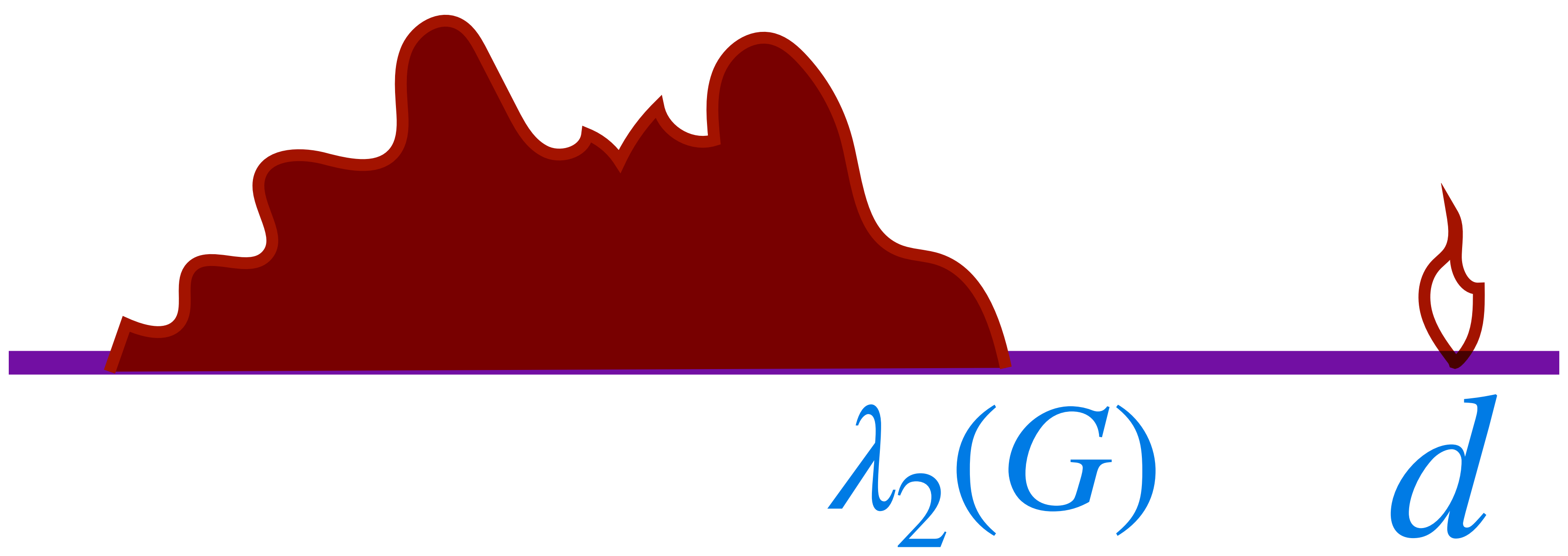

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Ramanujan graphs

Best spectral expanders

$\lambda_2(G) \leq 2\sqrt{d-1}$

Spectral expansion

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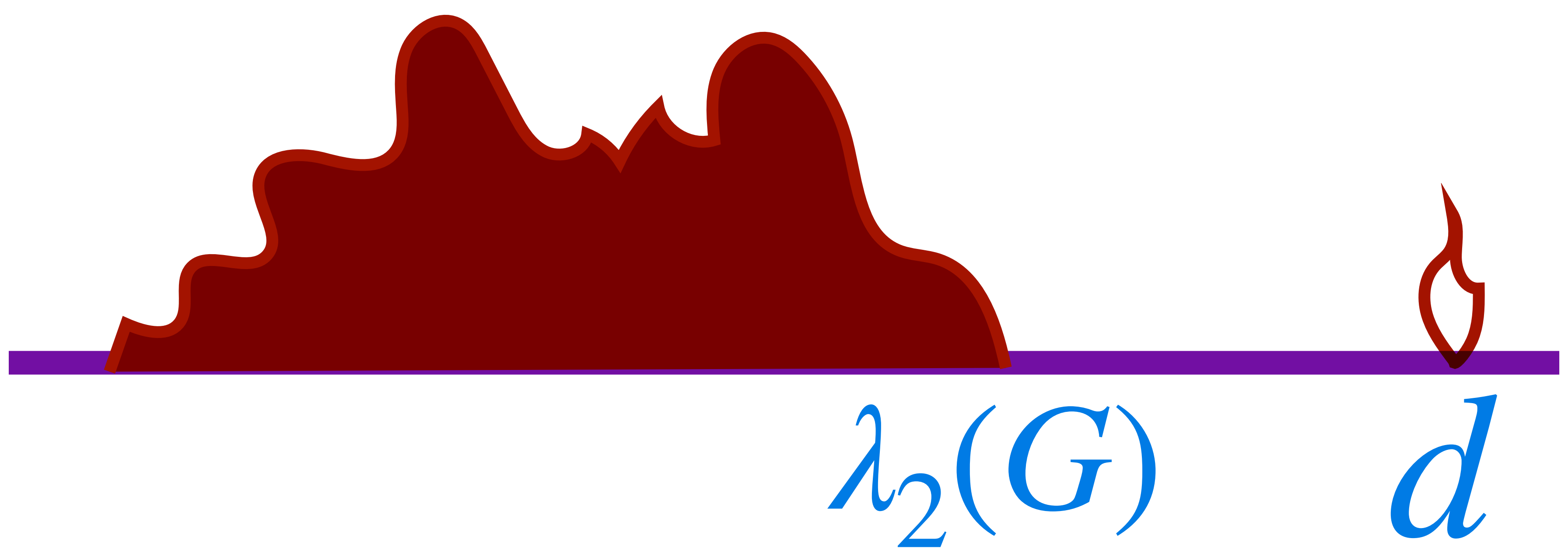

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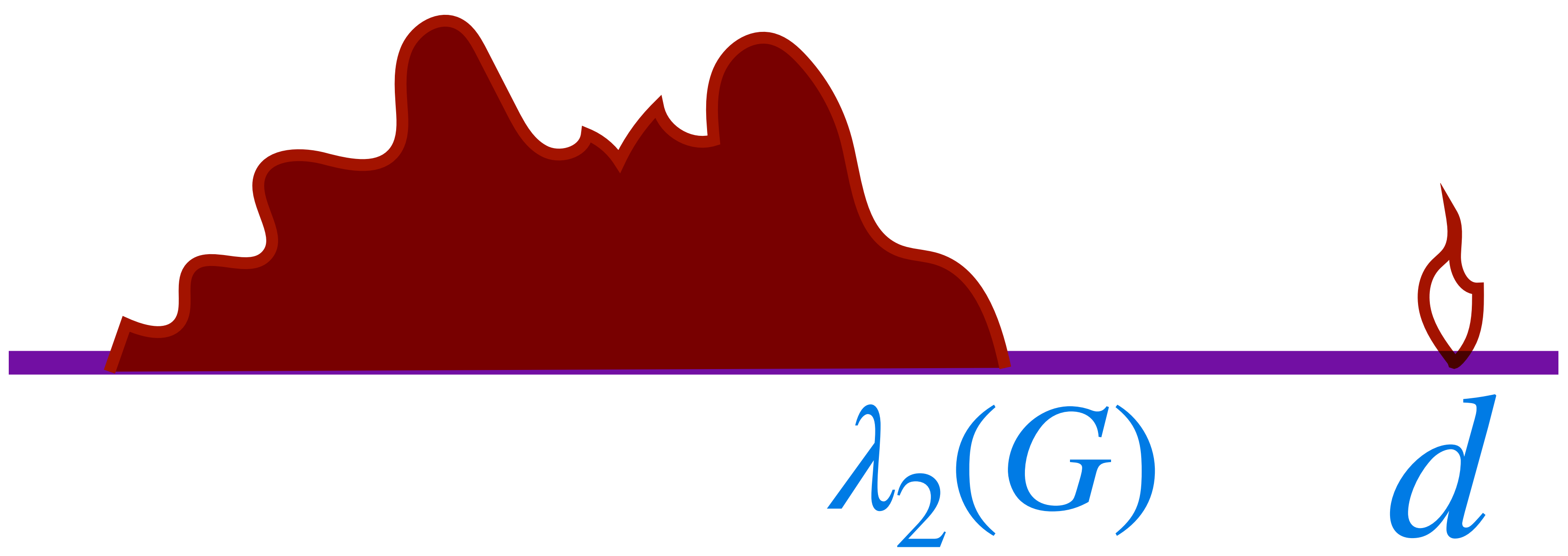

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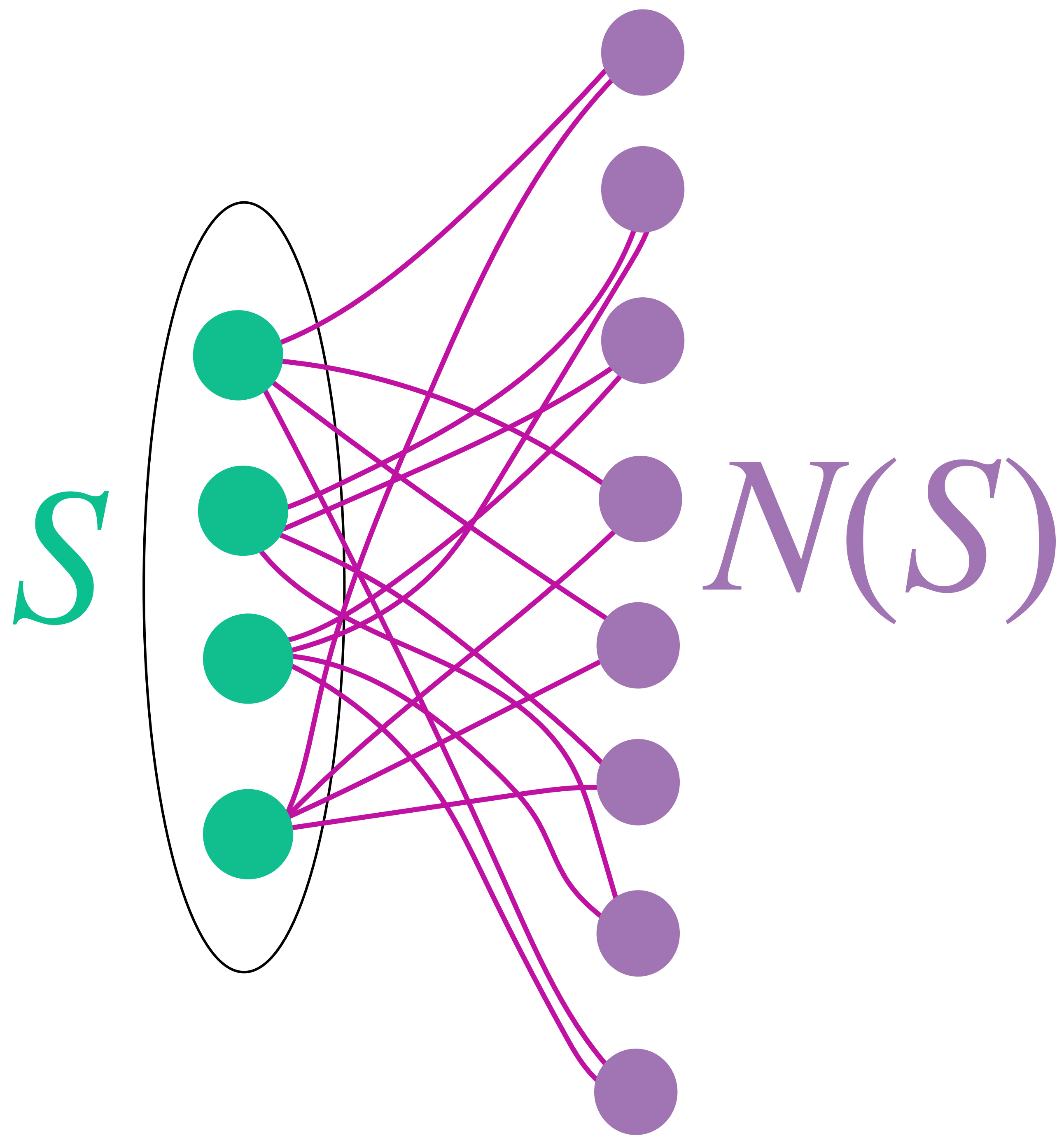
Spectral vs. small-set vertex expansion **[Kahale'95]**
Small-set vertex expansion in **Ramanujan graphs** $\geq d/2$



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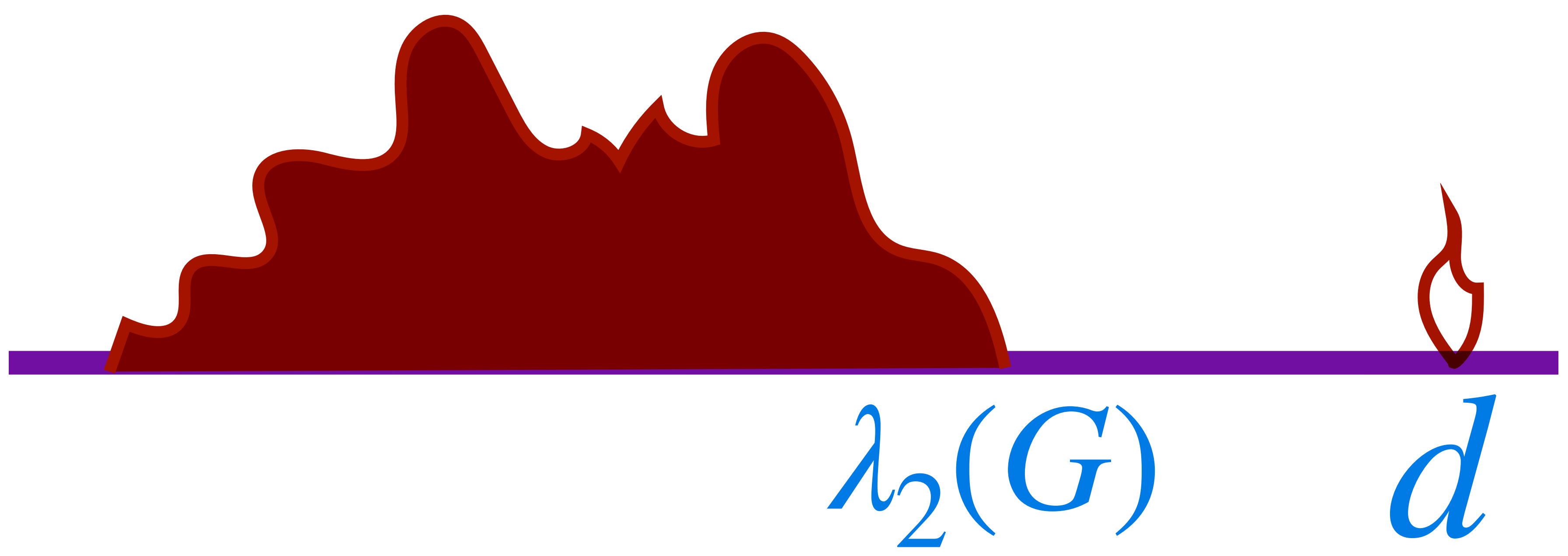

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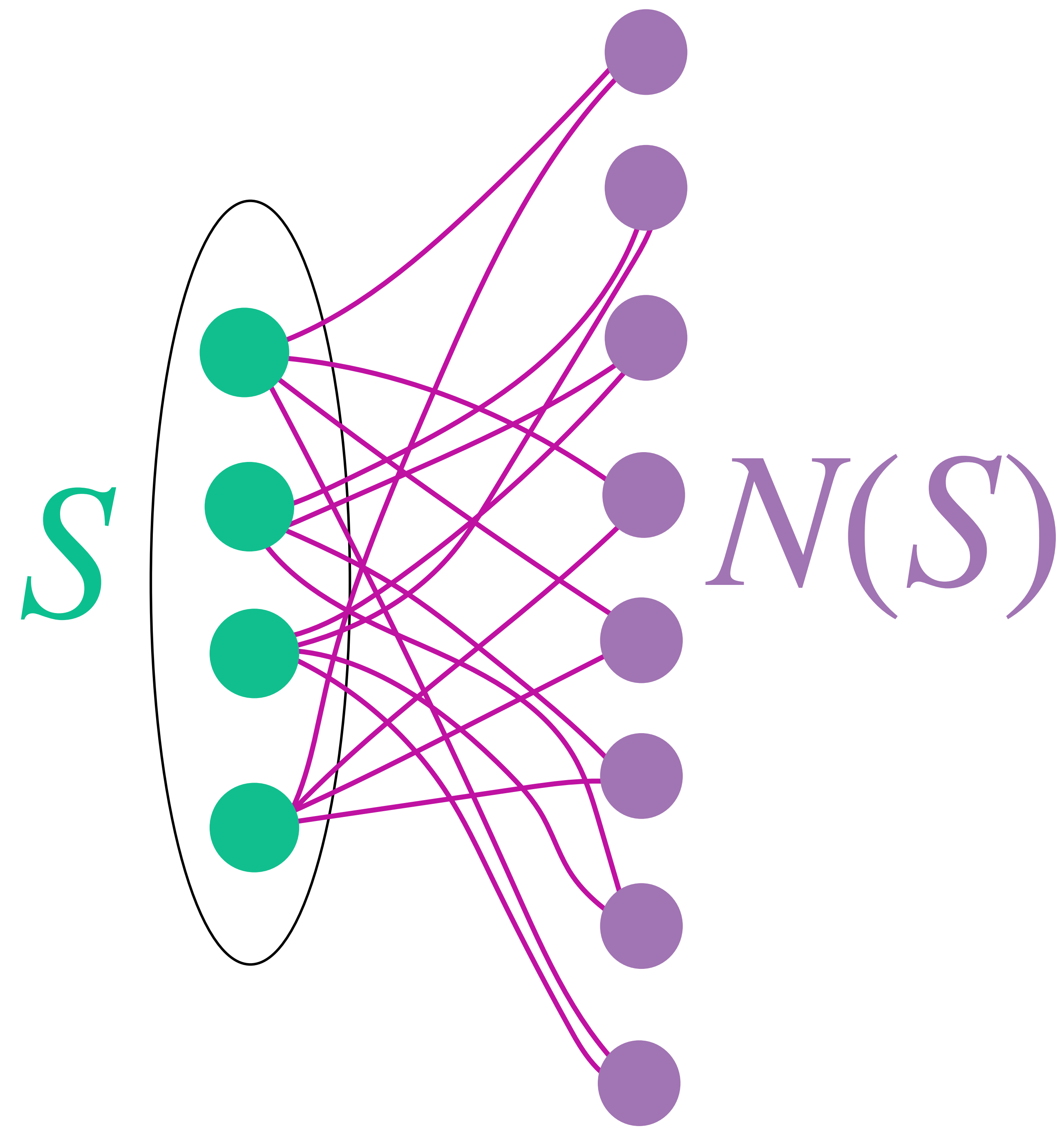
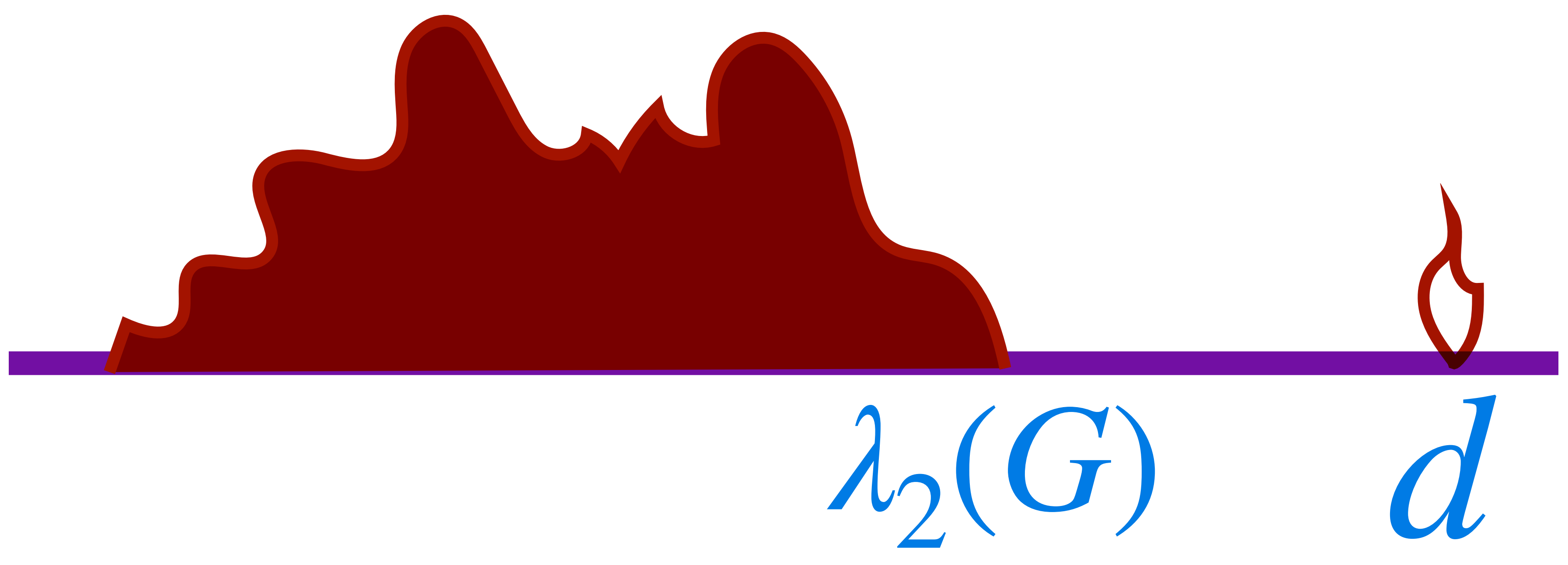
Barely falls short of UNE!



Spectral expansion $\not\Rightarrow$ Unique-neighbor expansion

Eigenvalues of adjacency matrix of G



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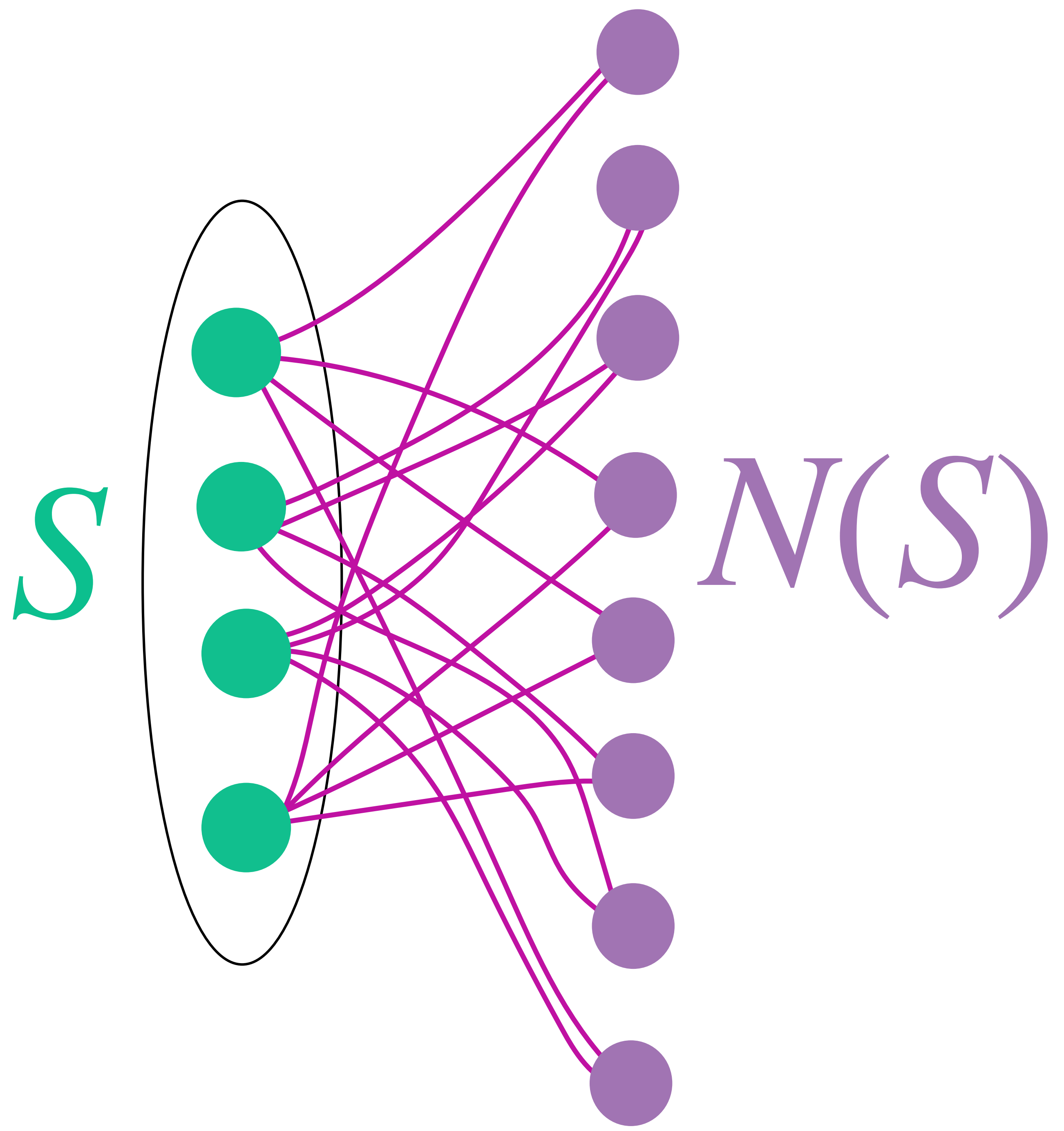


∃ Ramanujan graphs with small sets having

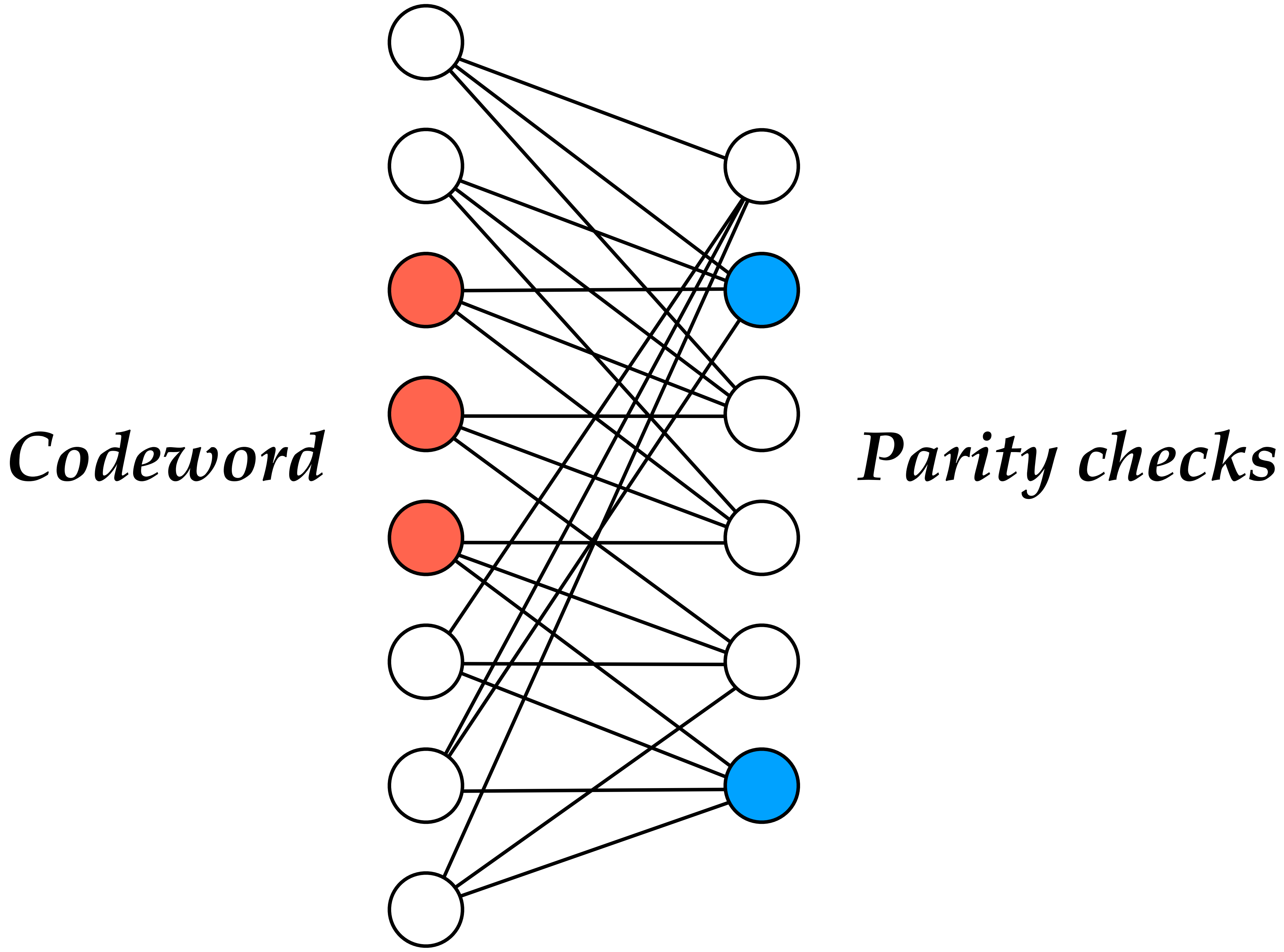
zero unique-neighbors

[Kahale'95, Mohanty-McKenzie'21, Kamber-Kaufman'22]!

Applications

LDPC codes

1-sided unique-neighbor expansion for δ-sized sets $\Longrightarrow$ distance $\geq \delta$.



Previous constructions

Explicit constructions [Alon-Capalbo'02, Capalbo-Reingold-Vadhan-Wigderson'02, Asherov-Dinur'23, Golowich'24]:

- 1-sided **lossless** expanders (having $(1-\varepsilon) \cdot d|S|$ unique-neighbors, not just $\Omega(d) \cdot |S|$).

- Any constant imbalance $|R|/|L|$.

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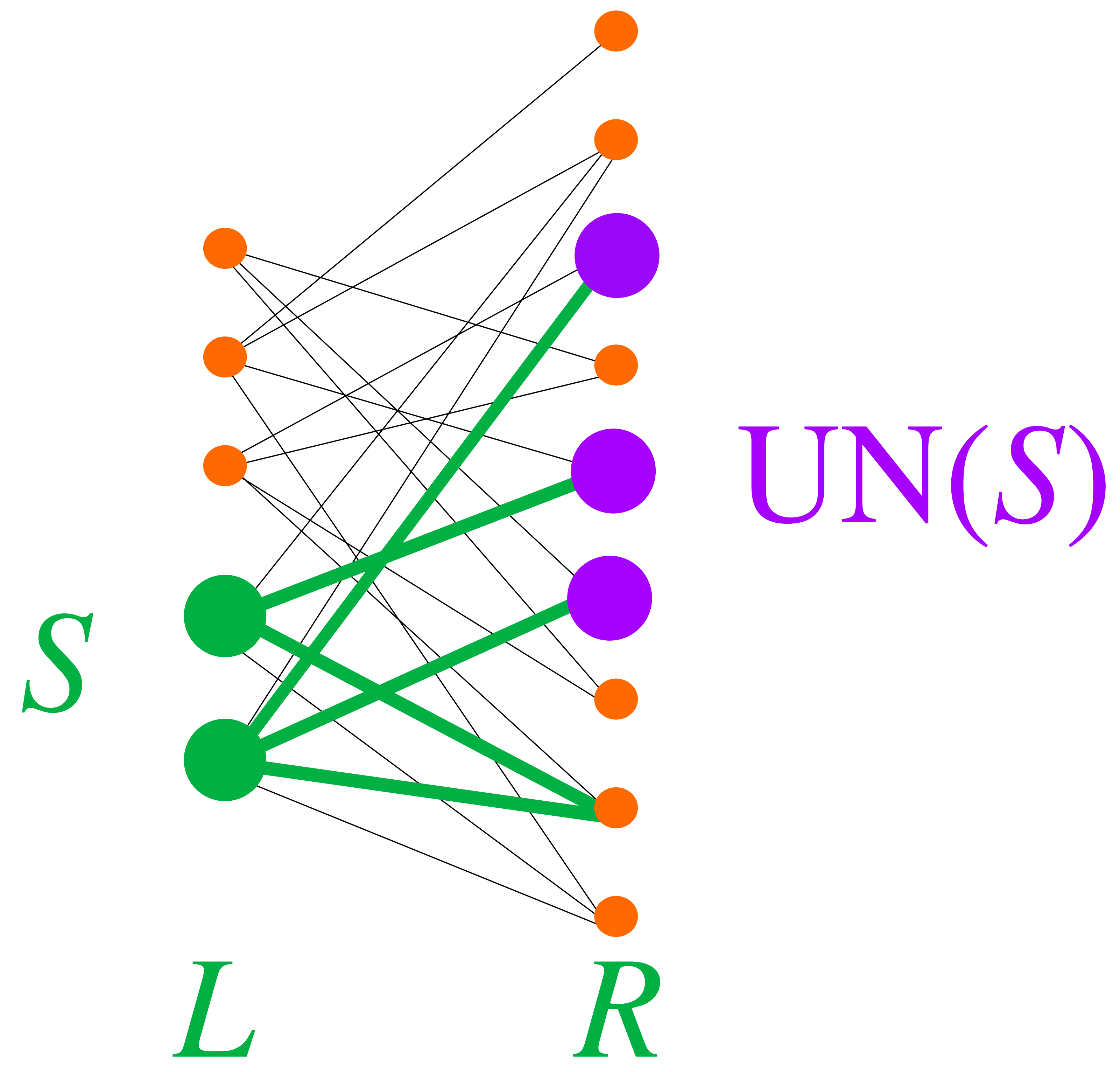
All previous explicit constructions are 1-sided only!

2-*sided* unique-neighbor expander

- For every $\delta|L|$-*sized* $S \subseteq L$
  $|\mathrm{UN}(S)| \geq \gamma \cdot d_1 |S|$
- For every $\delta|R|$-*sized* $S \subseteq R$
  $|\mathrm{UN}(S)| \geq \gamma \cdot d_2 |S|$

$\delta, \gamma > 0$ constants
d_1 left-degree, d_2 right-degree



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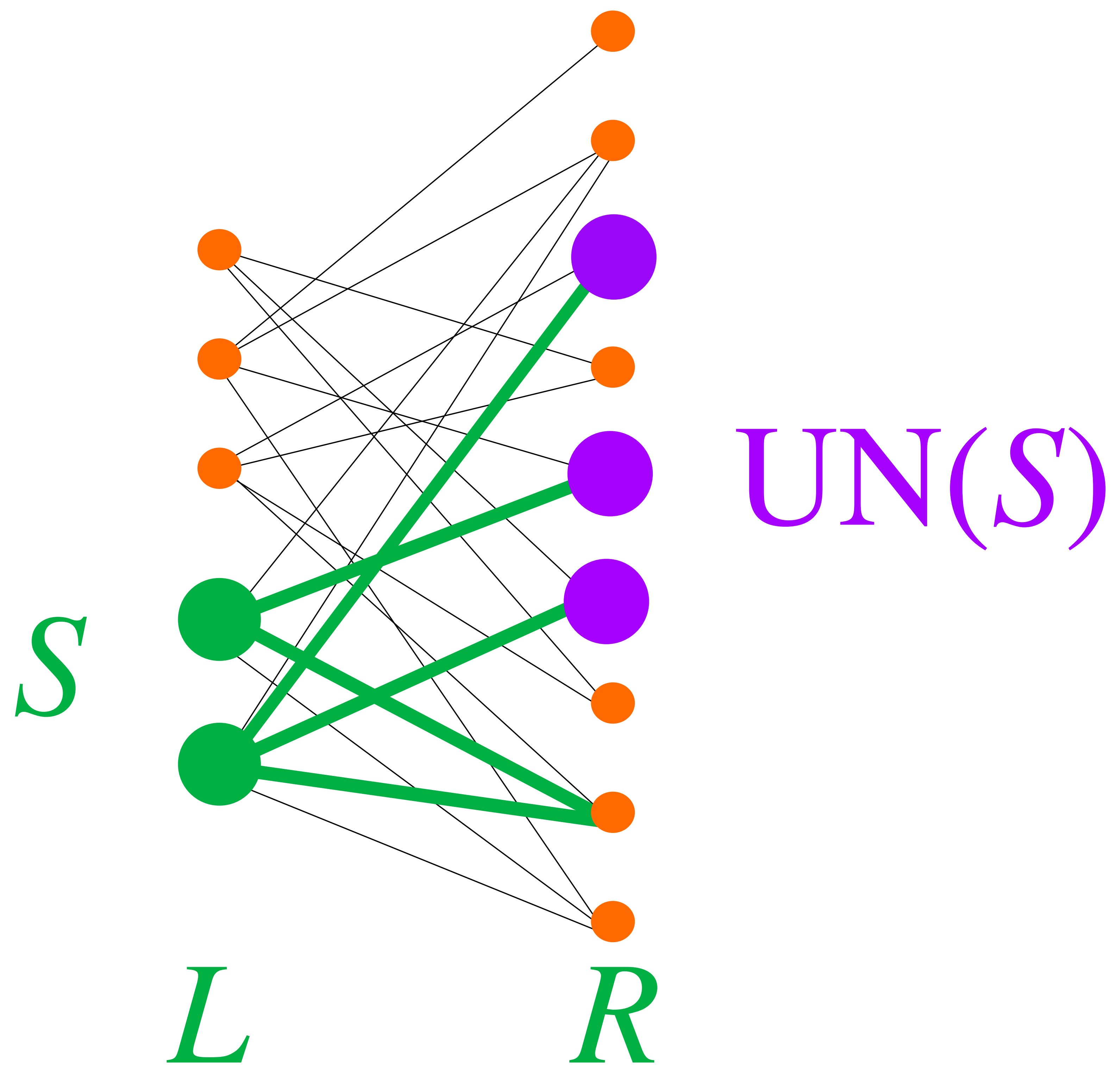

This work: ***explicit constructions*** of ***2-sided*** bipartite unique-neighbor expanders with arbitrary ***balance*** $|R|/|L|$.

Motivation for 2-sided expanders

Random graphs have this property, can we derandomize?

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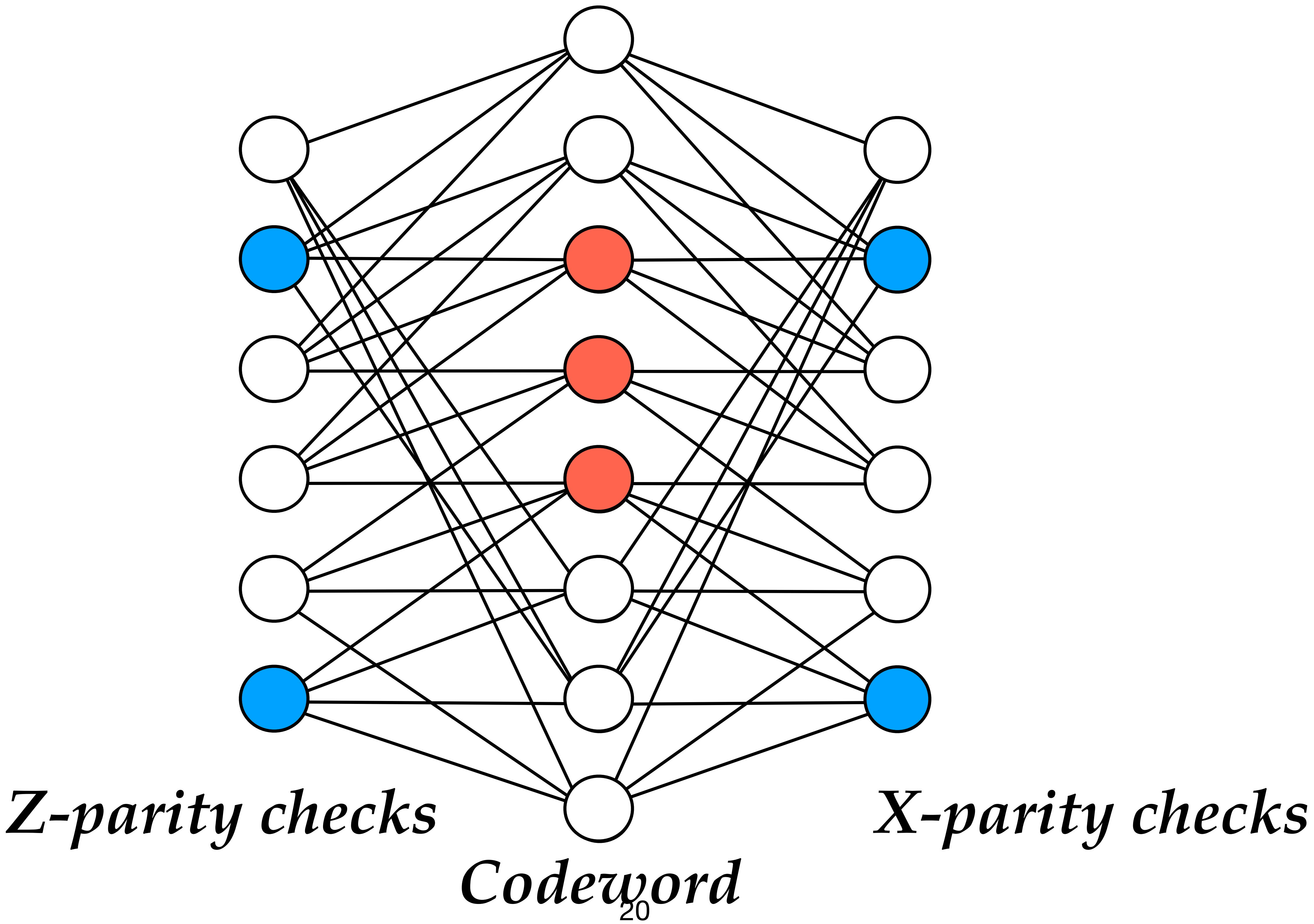
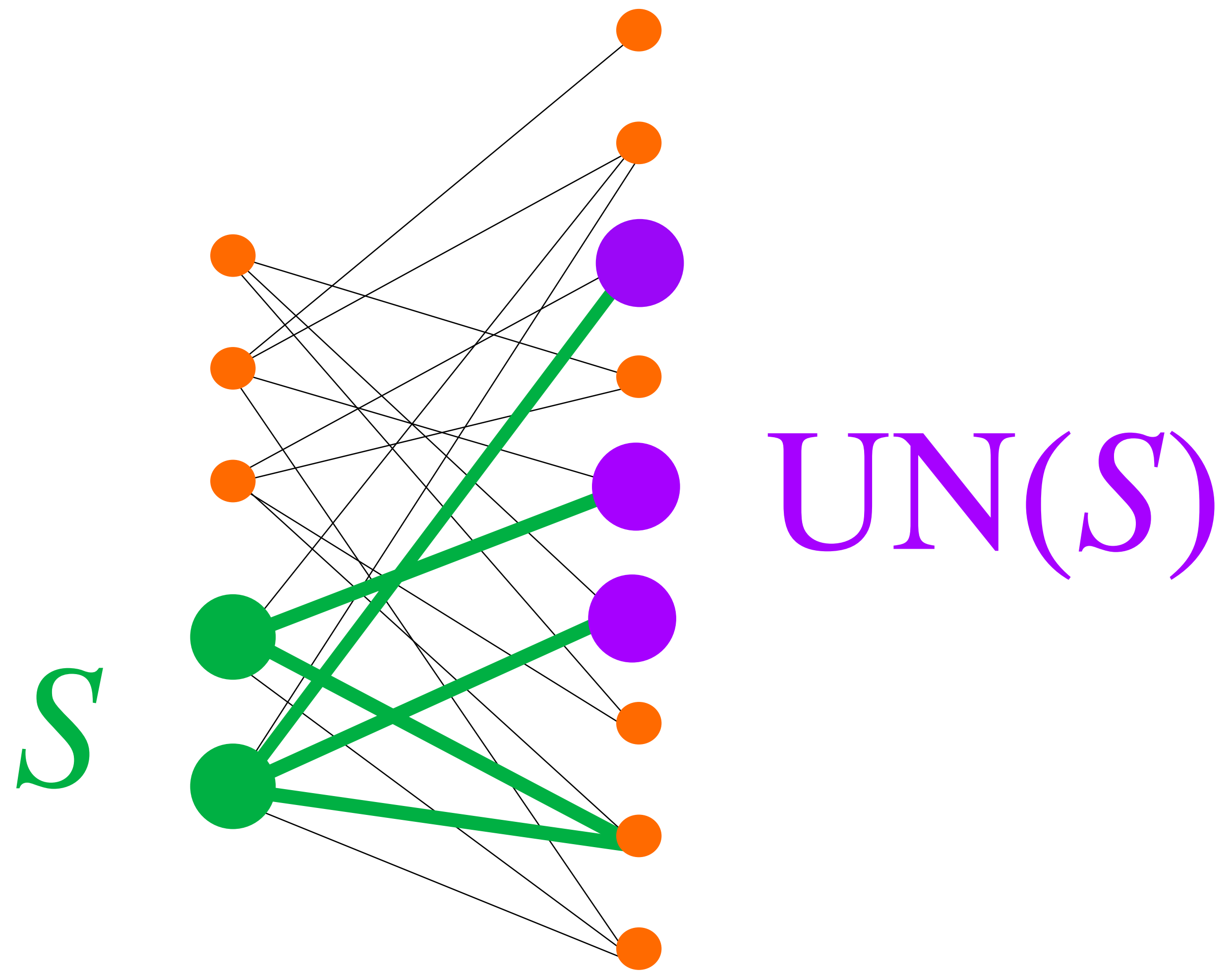
Quantum LDPC codes [Lin-M. Hsieh'22]:

2-sided "algebraic" lossless expanders $\Longrightarrow$ efficiently decodable quantum LDPC codes!



Our results

For $d_1,\ d_2$ large enough: explicit infinite family of
$(d_1,\ d_2)$-biregular 2-*sided* unique-neighbor expanders



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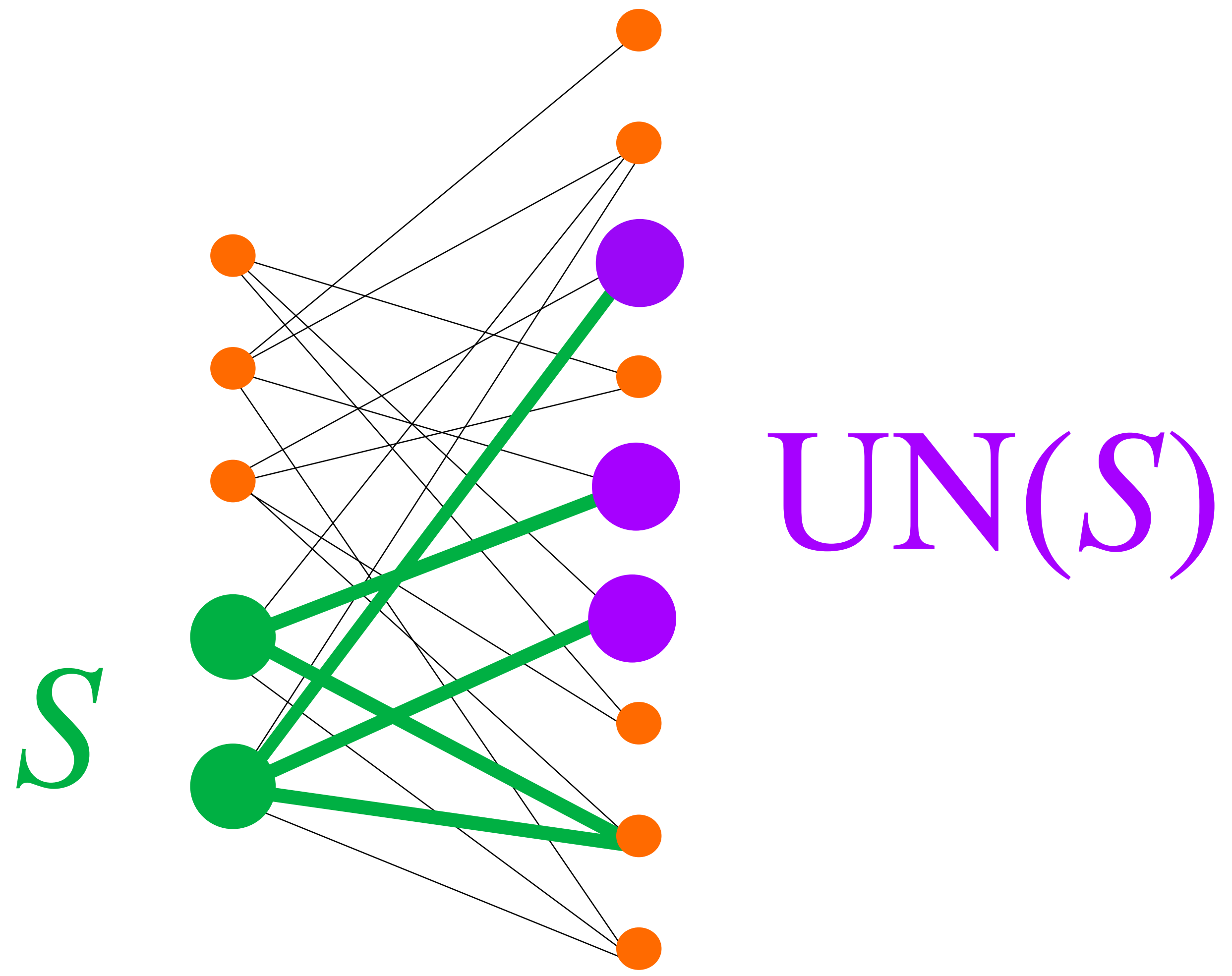


$|\mathrm{UN}(S)| \geq \Omega(d) \cdot |S|$ when $|S| \leq \delta n$

In addition, **small sets expand losslessly.**

$$|\mathrm{UN}(S)| \geq (1-\varepsilon) \cdot d\,|S|$$

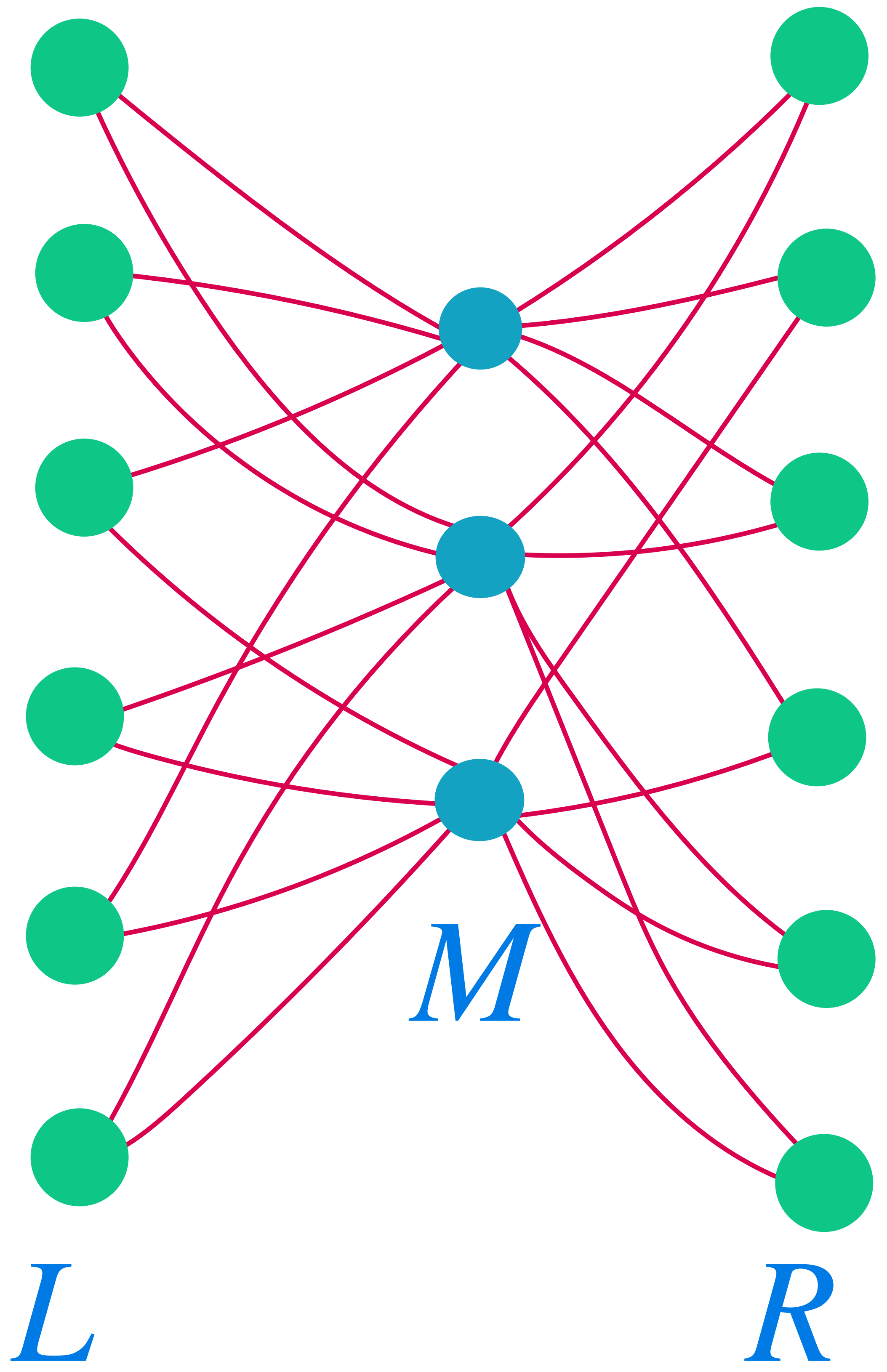
when $|S| \leq \exp\left(O\left(\sqrt{\log n}\right)\right)$.

Construction

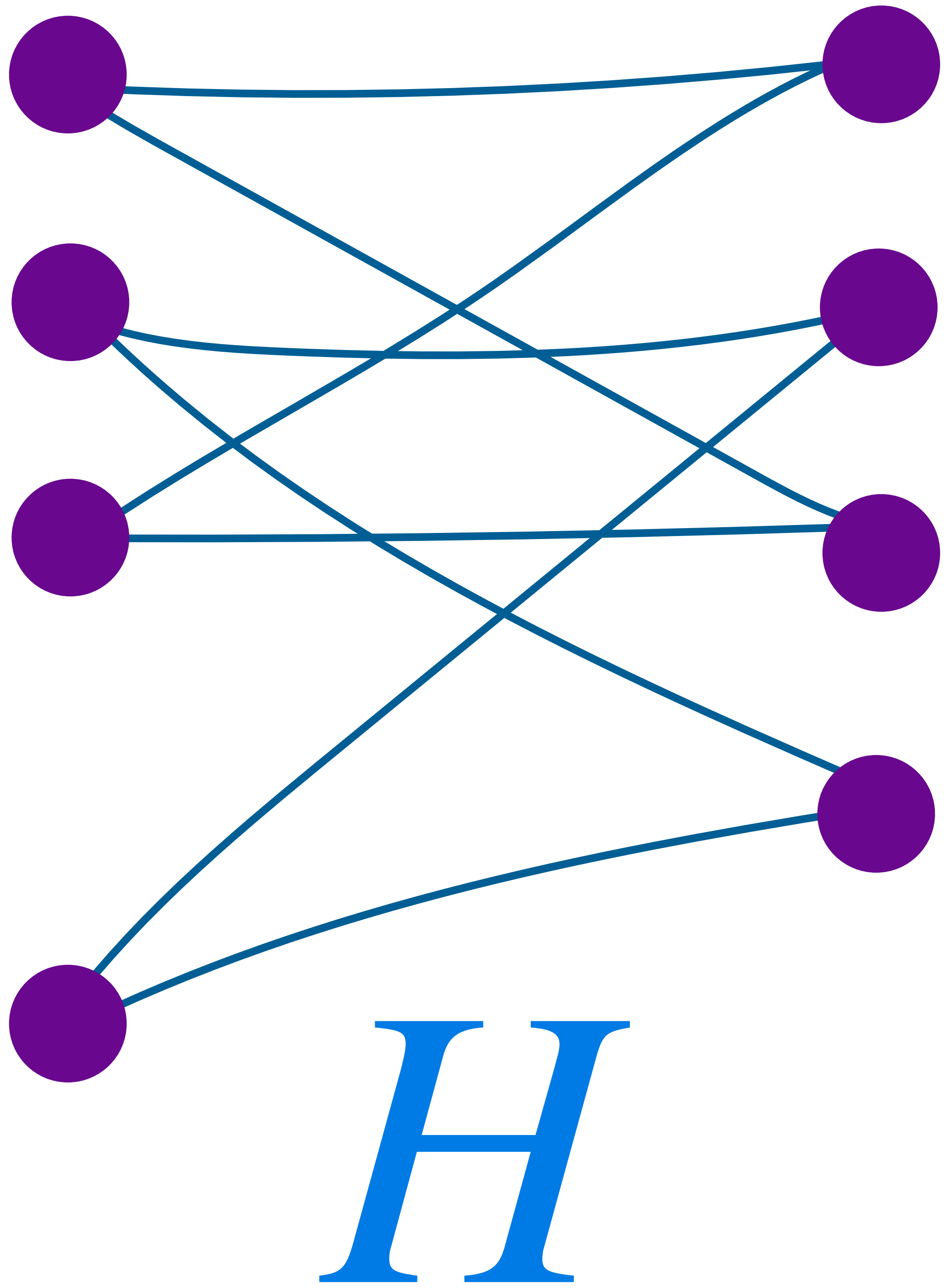
Tripartite Line Product

Base graph
LM, MR bipartite spectral expanders
(# vertices $\rightarrow \infty$)

Gadget graph
Bipartite lossless expander
(O(1)-sized random graph)



$\times$



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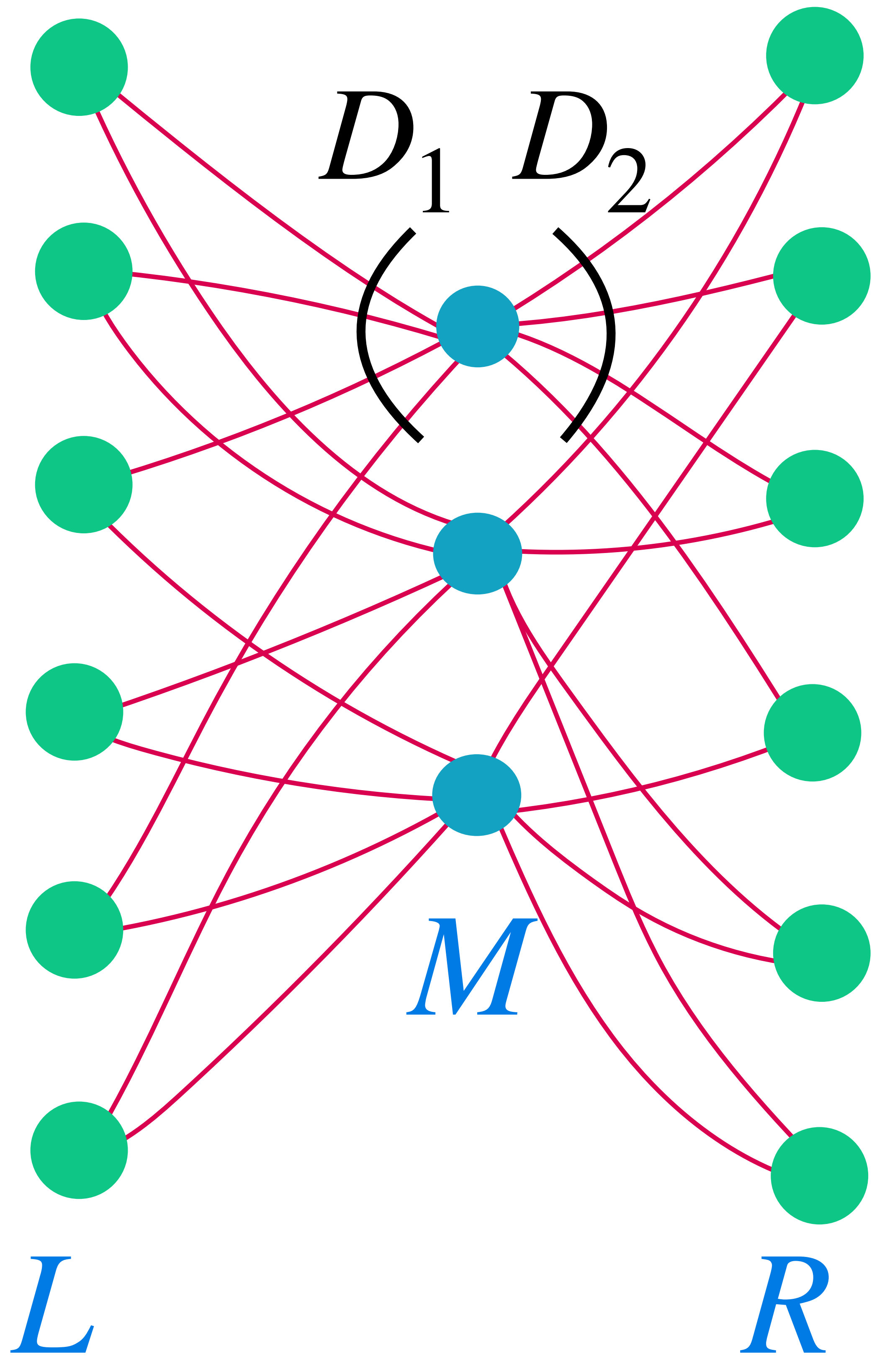

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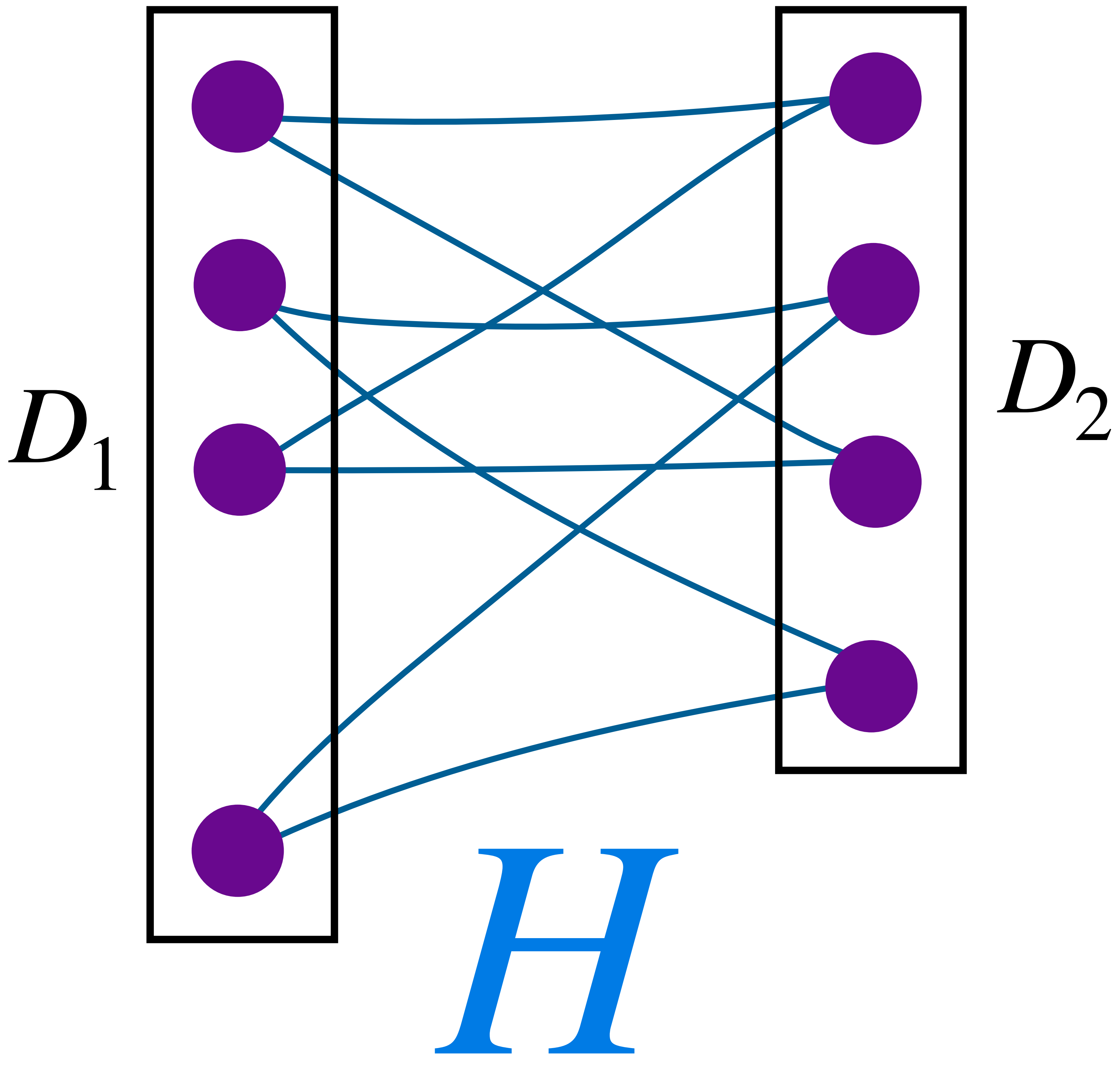
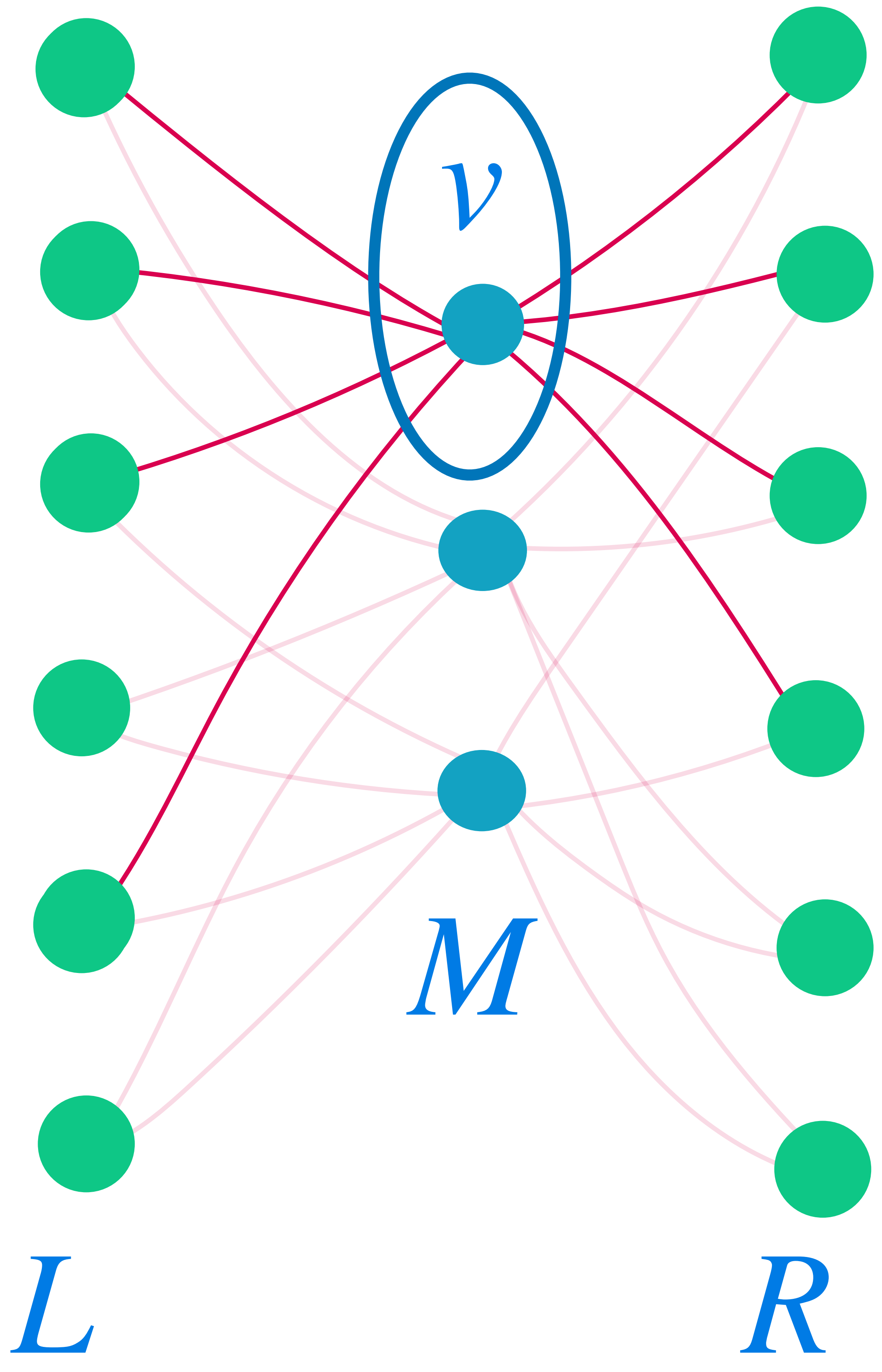


Construction

Tripartite Line Product

For each $v \in M$, place copy of gadget H between $N_L(v)$ and $N_R(v)$



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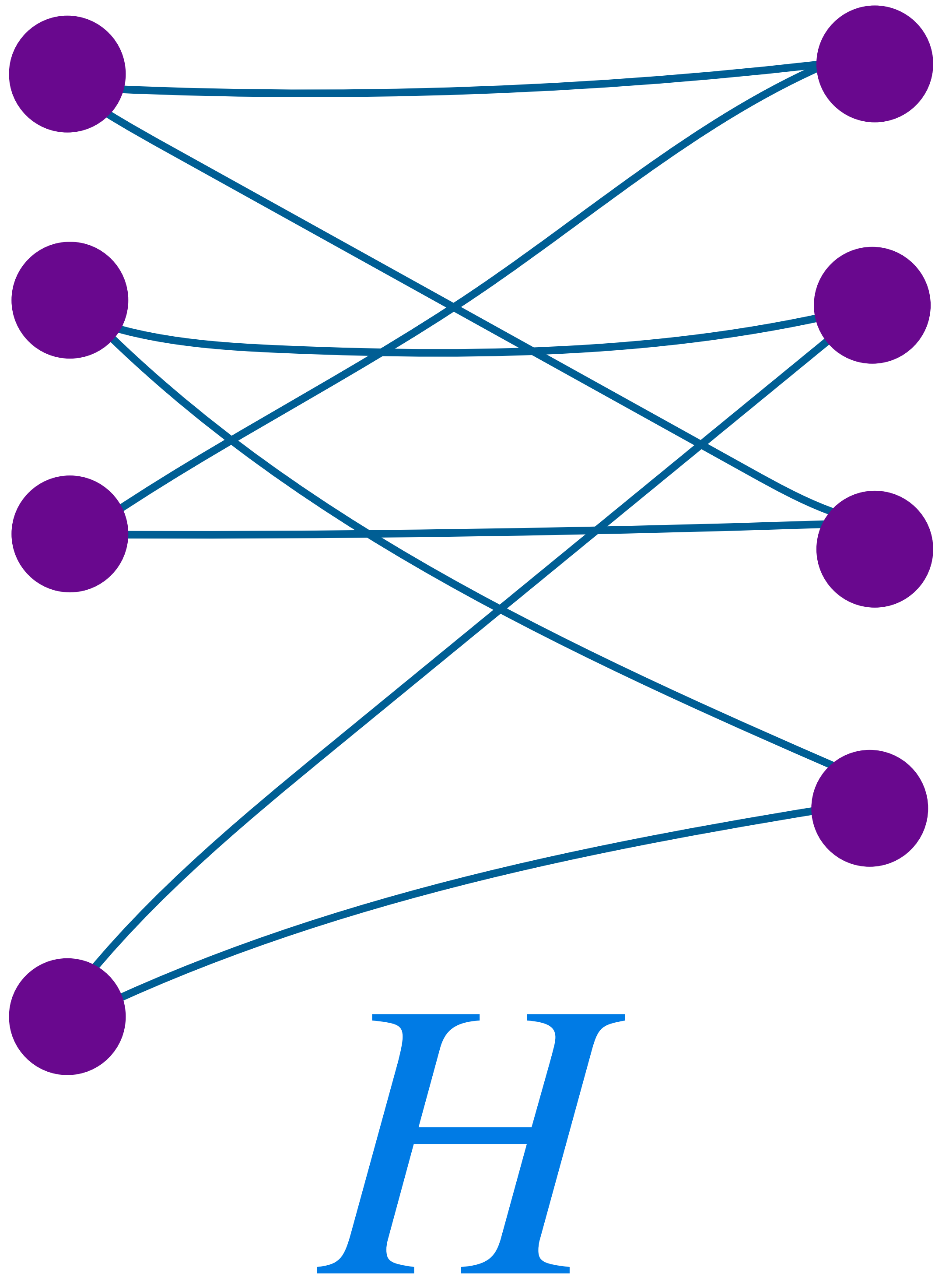
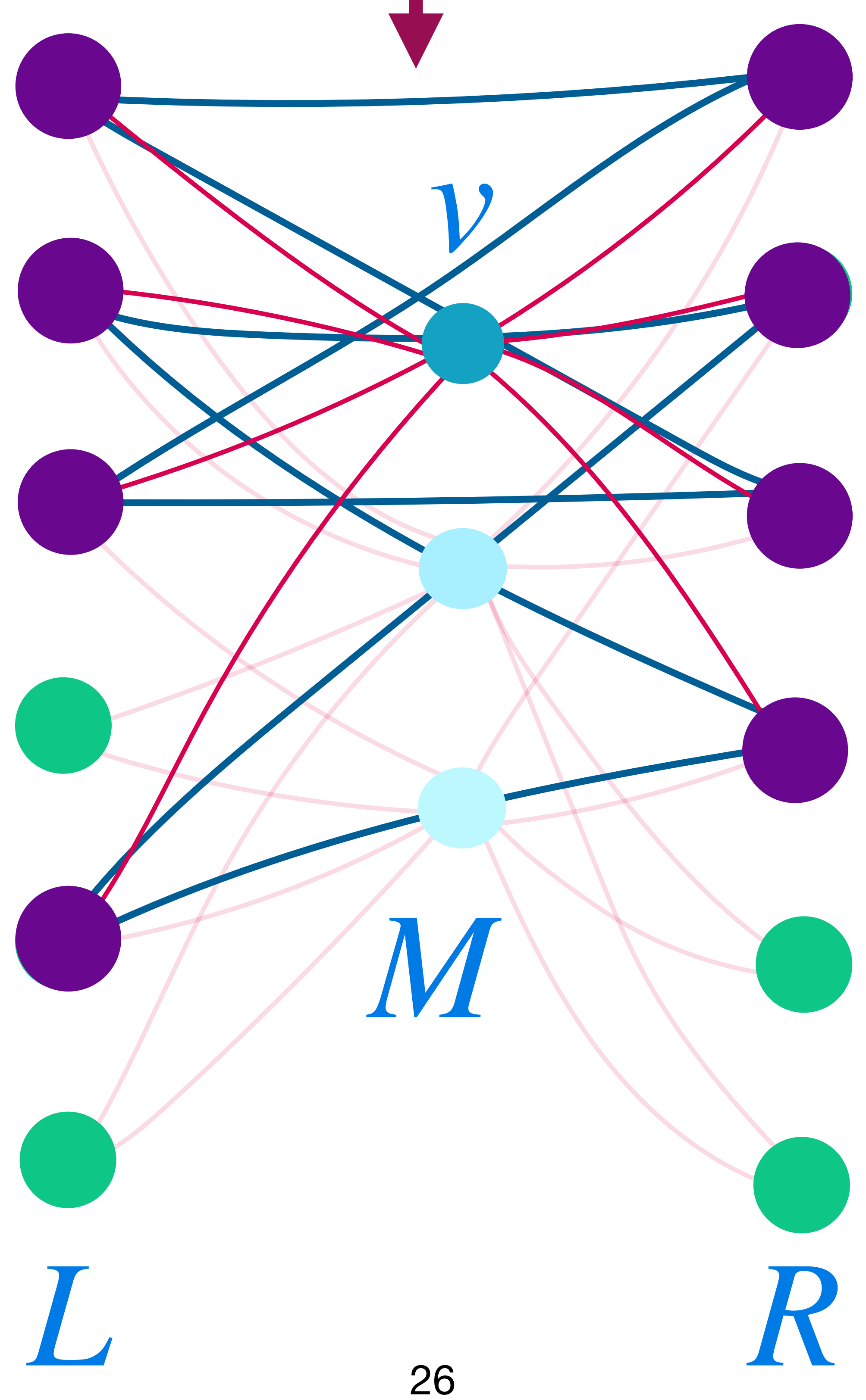


Construction

Tripartite Line Product

Copy of H between $N_L(v)$ and $N_R(v)$



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Tripartite Line Product

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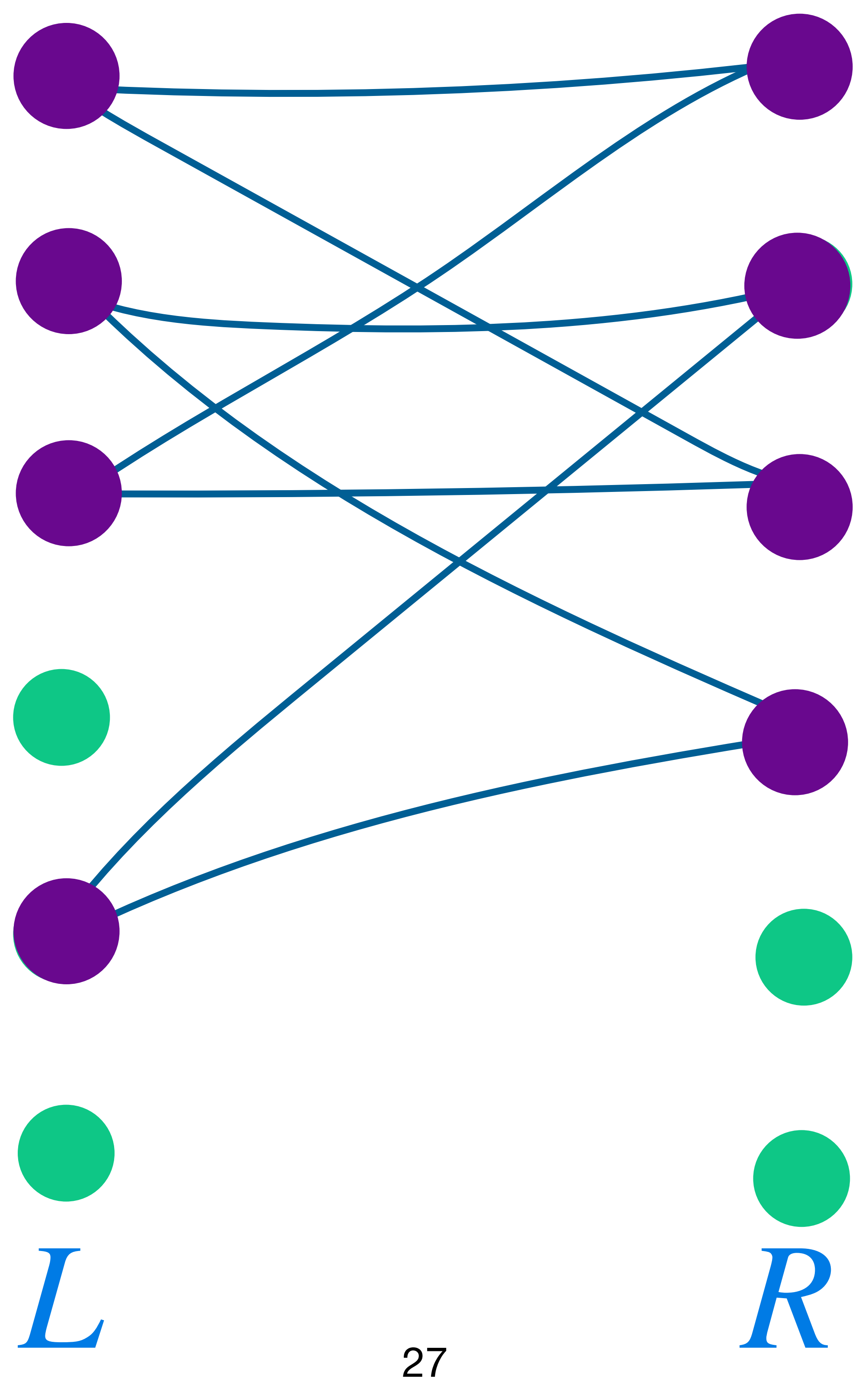


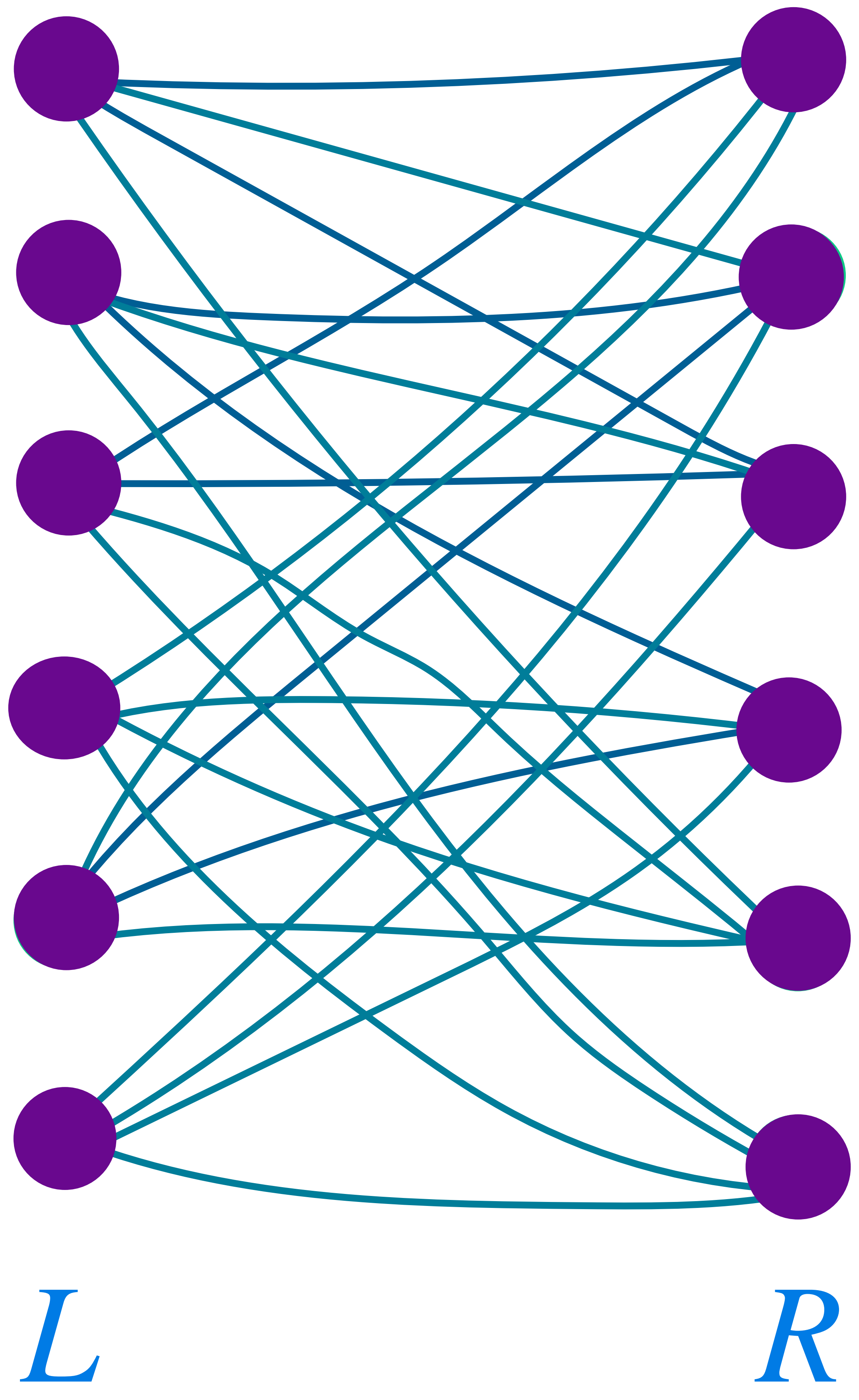
Construction

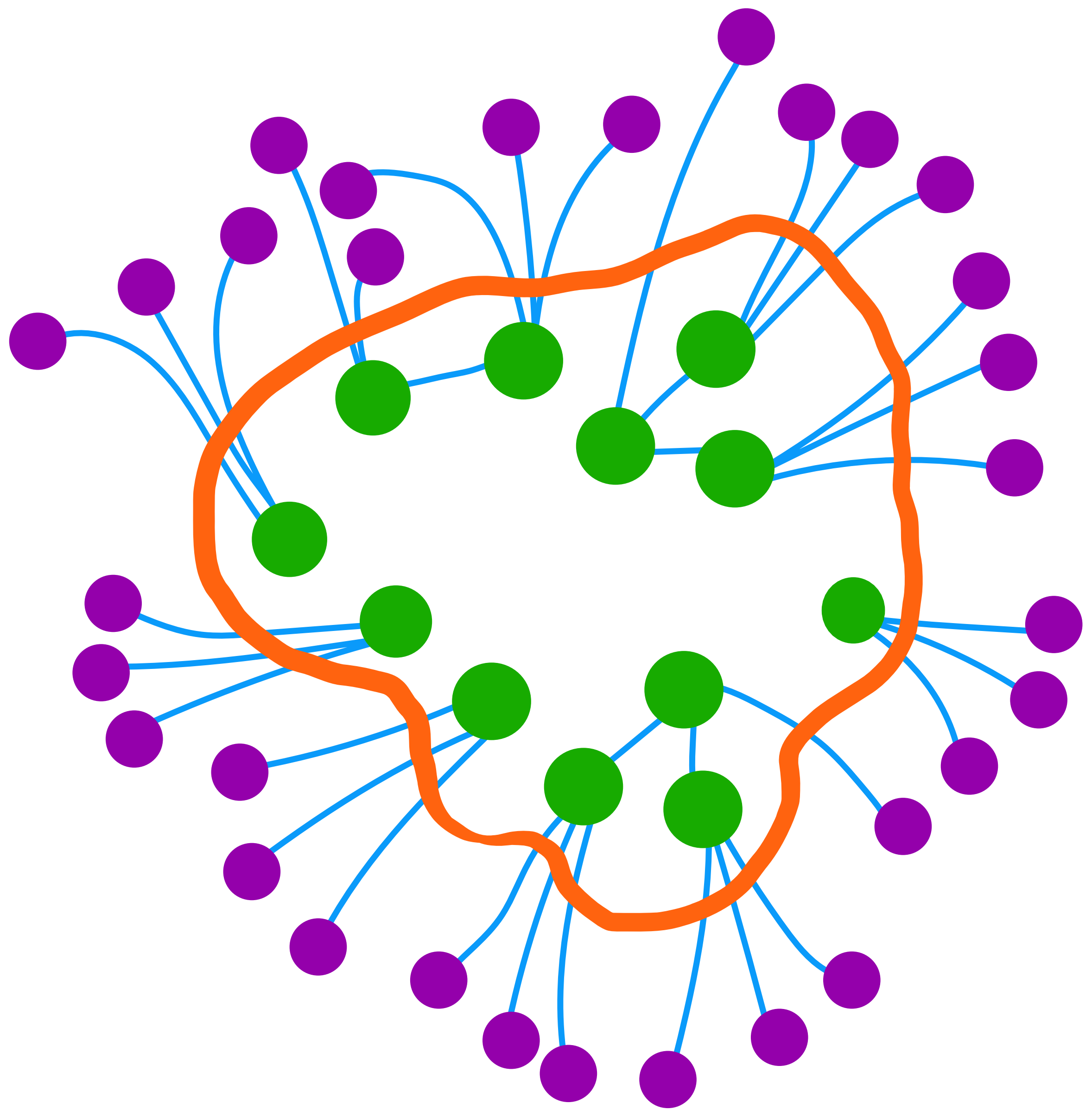
Tripartite Line Product

After placing all copies

$$|\mathrm{UN}(S)| \geq \Omega(d) \cdot |S|$$

when $|S| \leq \delta n$





$$|\mathrm{UN}(S)| \geq (1-\varepsilon) \cdot d|S|$$

when $|S| \leq \exp\left(O\left(\sqrt{\log n}\right)\right)$

New results in spectral graph theory

Near-Ramanujan (d_1, d_2)-biregular graphs:

$$\lambda_2 \leq \left(\sqrt{d_1 - 1} + \sqrt{d_2 - 1}\right) \cdot (1 + \varepsilon)$$

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Theorem. For a near-Ramanujan (d_1, d_2)-biregular graph G and any $S_1 \subseteq L$, $S_2 \subseteq R$ of "linear" size, the left and right average degrees $\tilde{d}_1$, $\tilde{d}_2$ of $G[S_1 \cup S_2]$ satisfy

$$(\tilde{d}_1 - 1)(\tilde{d}_2 - 1) \leq \sqrt{(d_1 - 1)(d_2 - 1)} \cdot (1 + O(\varepsilon)).$$

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Upper bound on the spectral radius of the **non-backtracking matrix** of $G[S_1 \cup S_2]$.

Thank you!

Questions?