A simple and sharper proof of the hypergraph Moore bound

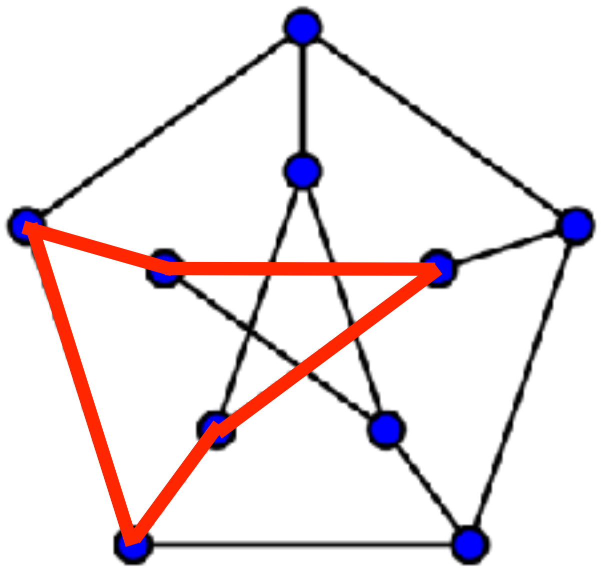
Jun-Ting (Tim) Hsieh
Carnegie Mellon

Pravesh K. Kothari
Carnegie Mellon

Sidhanth Mohanty
UC Berkeley

Girth of a graph

Girth: length of the shortest cycle.



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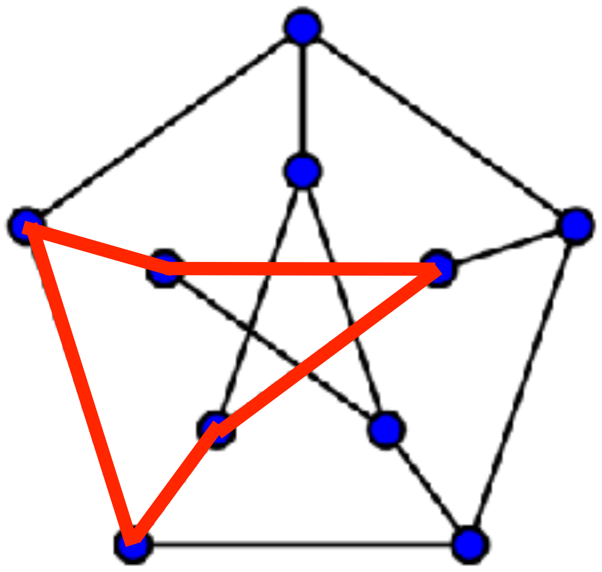

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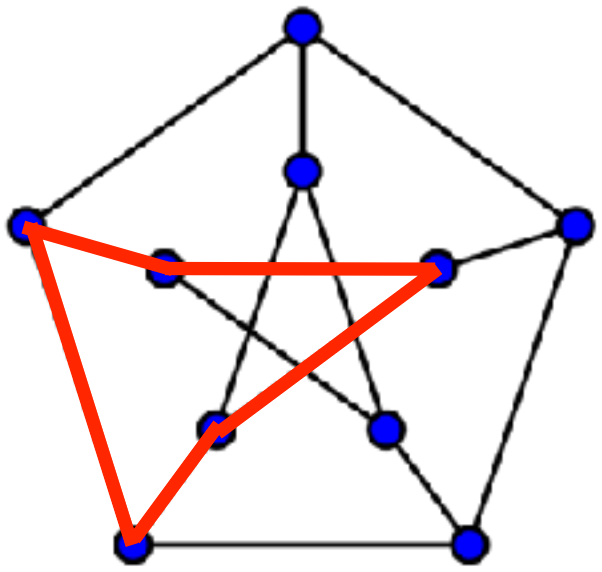

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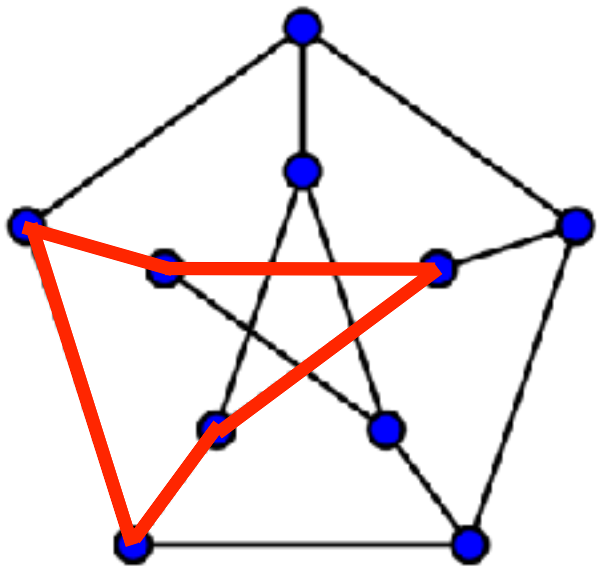

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- What about $d = 2.1$?

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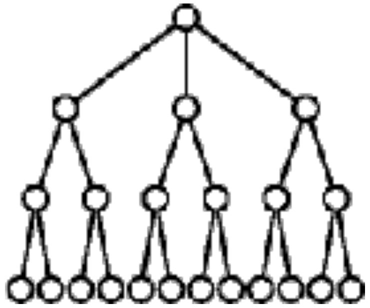

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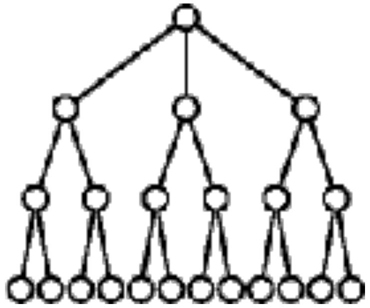

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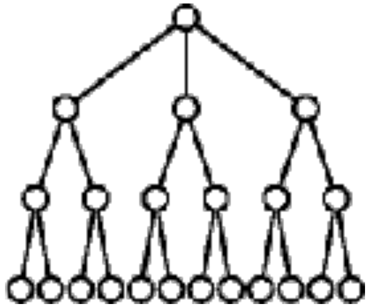

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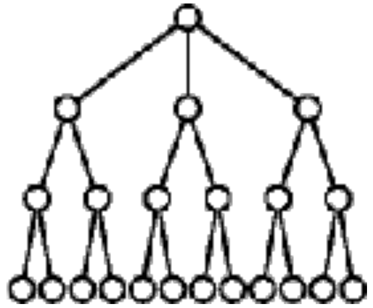

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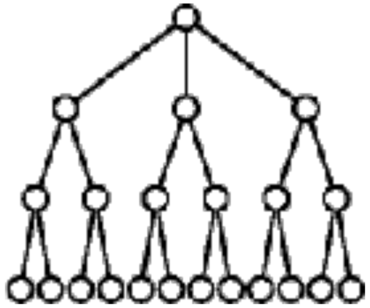

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Bollabas [1978] asked: **irregular** graphs with **average** degree d ?

Classical Moore bound

Alon, Hoory and Linial [2002] proved that $g \leq 2\log_{d-1} n + 2$ holds for **irregular** graphs with **average** degree d.

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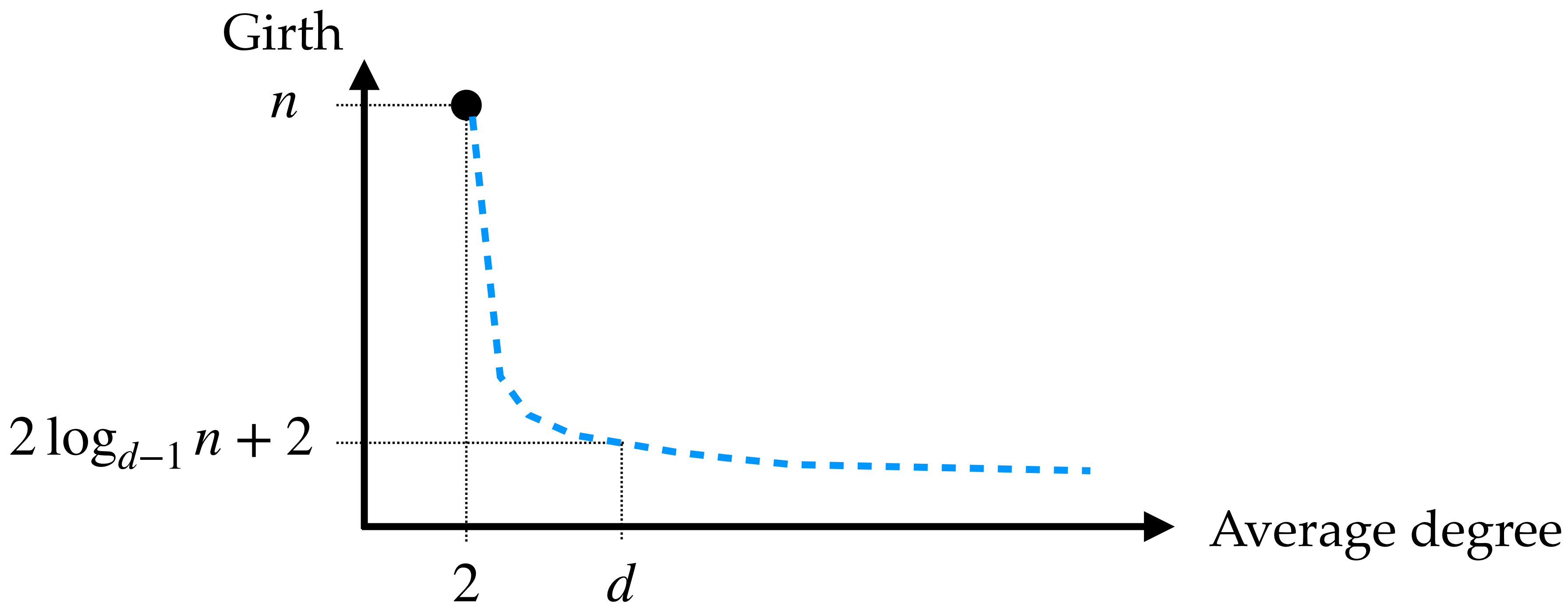

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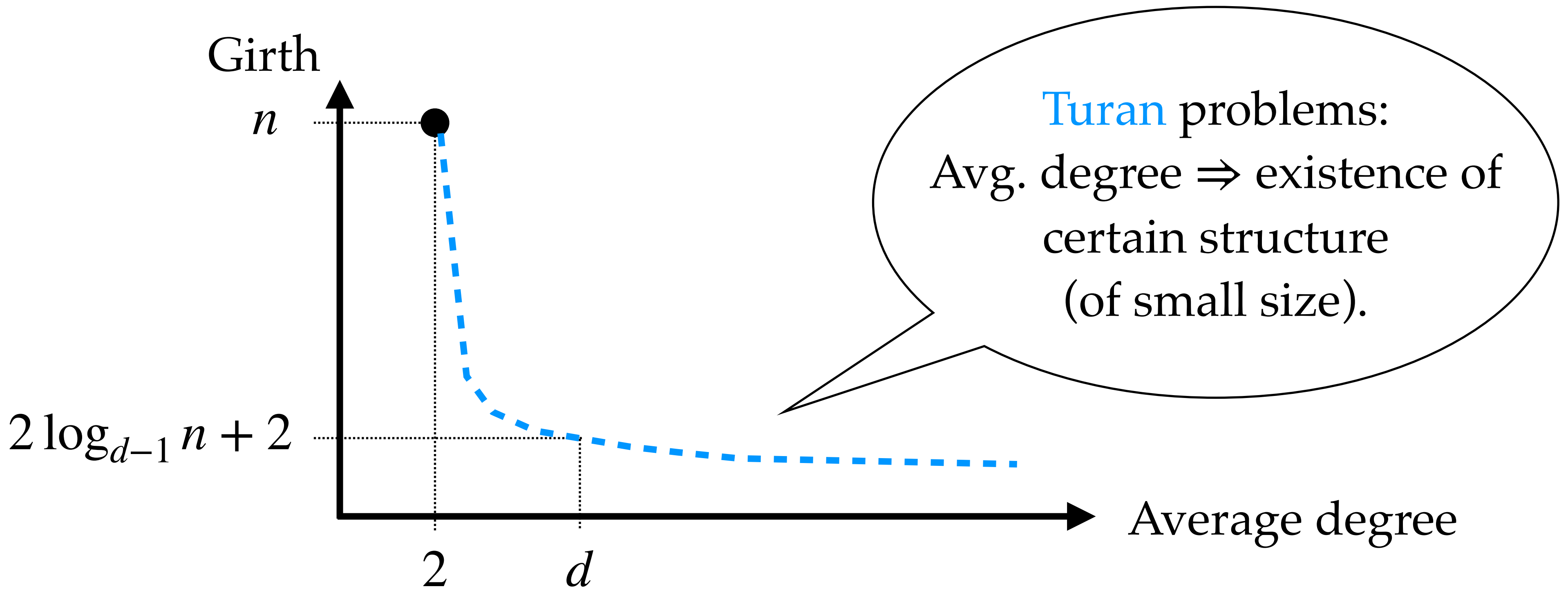

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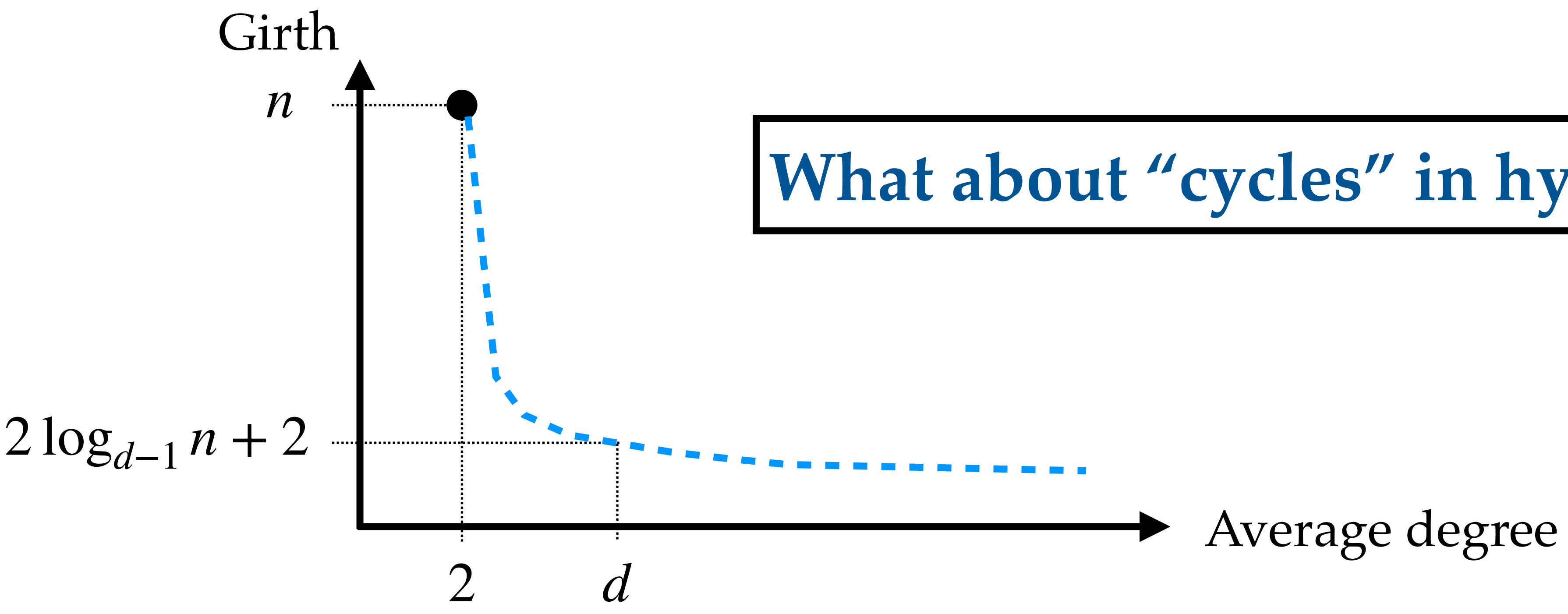
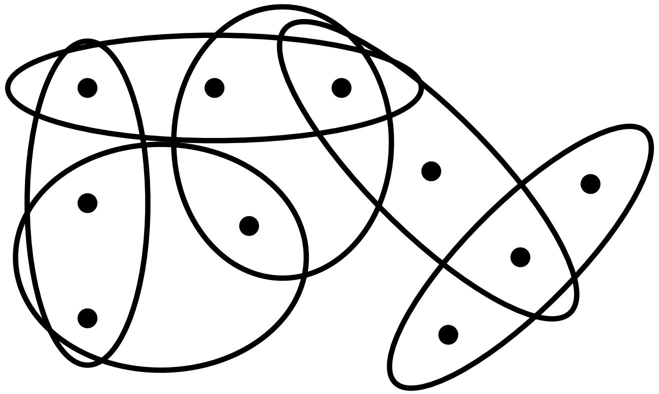


What about "cycles" in hypergraphs?

Hypergraphs

A *k*-uniform hypergraph is just a graph but each hyperedge has *k* vertices.



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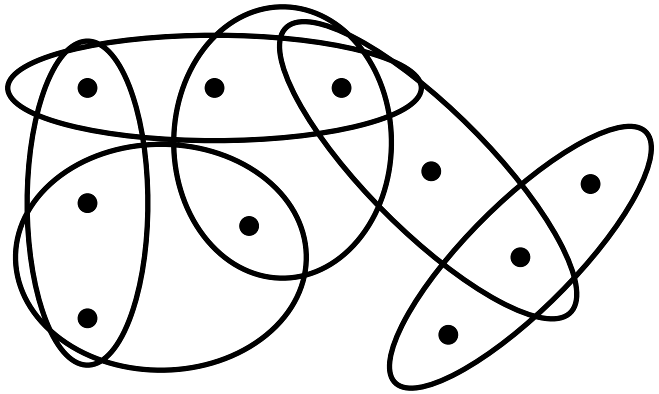
A **cycle** in a hypergraph (a.k.a. **even cover**) is a set of hyperedges such that every vertex participates in an **even** number of them.



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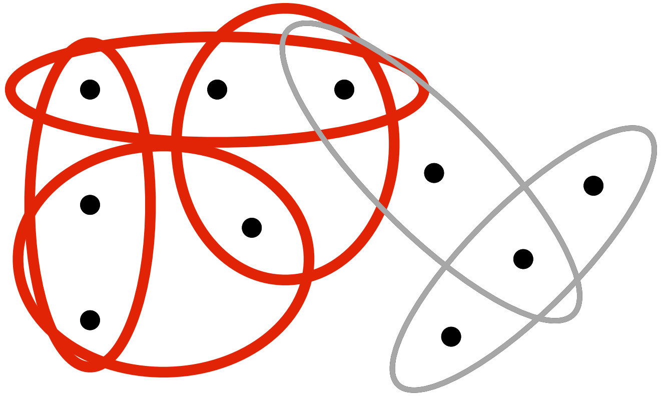

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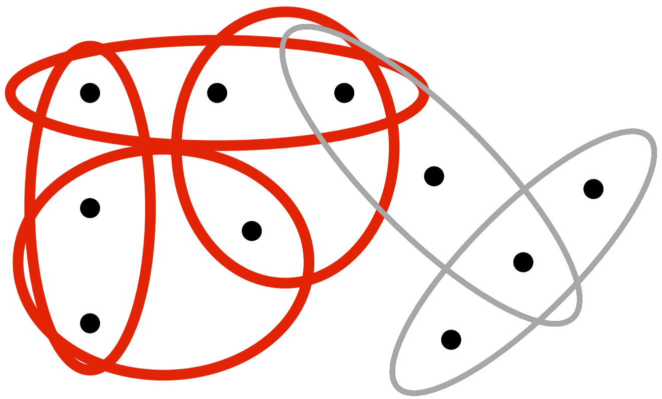
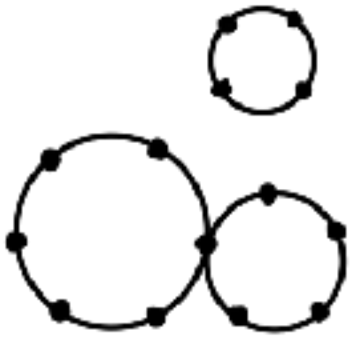




For graphs:
Even cover ⇔ union of cycles

Even covers

Another view:

Even covers

Another view:

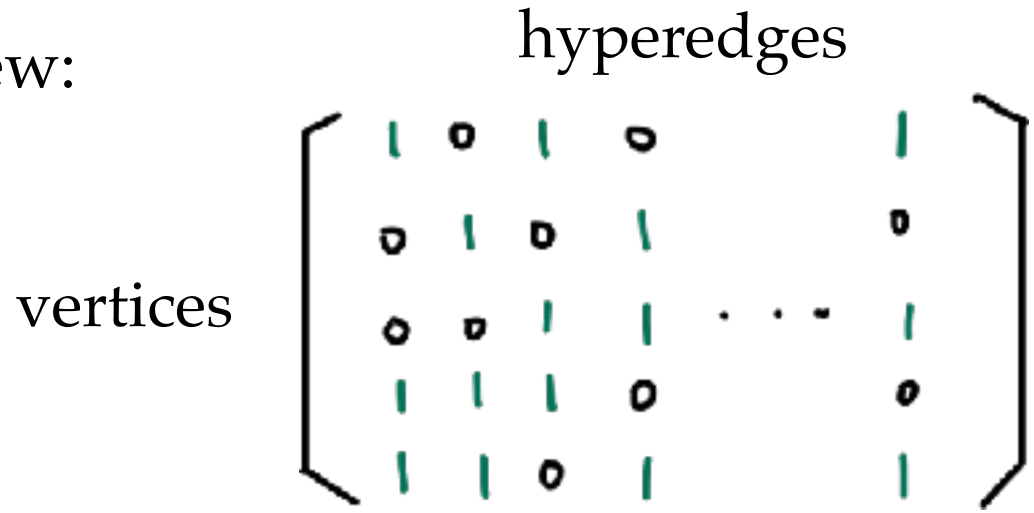
hyperedges

vertices

$$\begin{bmatrix} 1 & 0 & 1 & 0 & & 1 \\ 0 & 1 & 0 & 1 & & 0 \\ 0 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & & 0 \\ 1 & 1 & 0 & 1 & & 1 \end{bmatrix}$$

Even covers

Another view:



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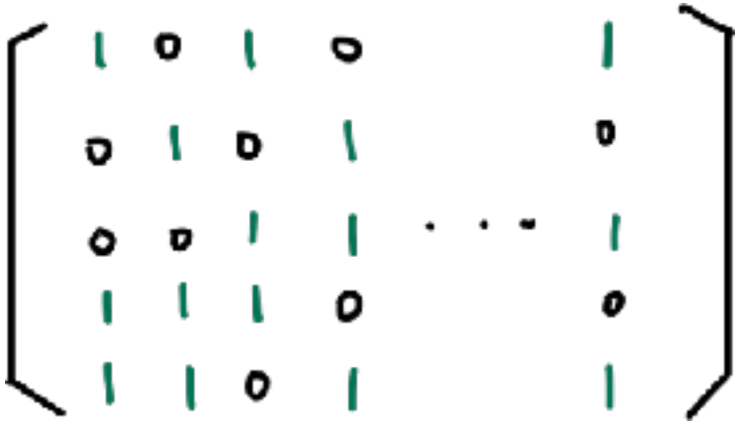

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Girth = size of a smallest linearly dependent subset of columns.

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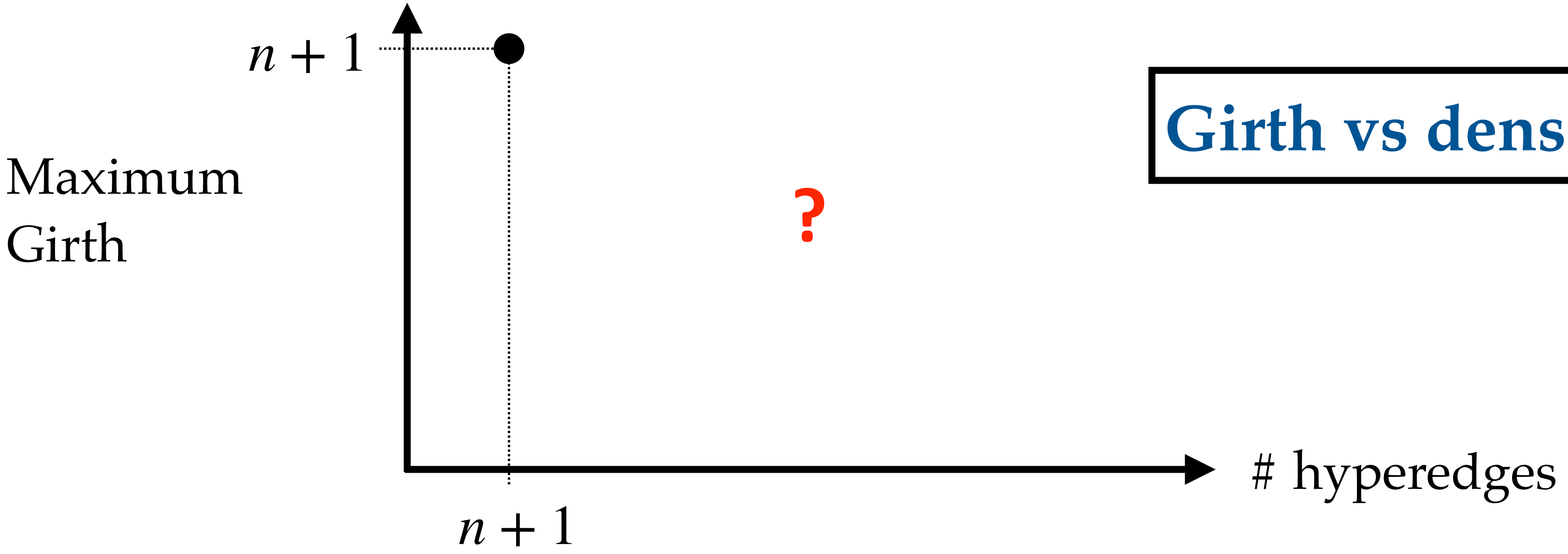

Girth vs density trade-offs

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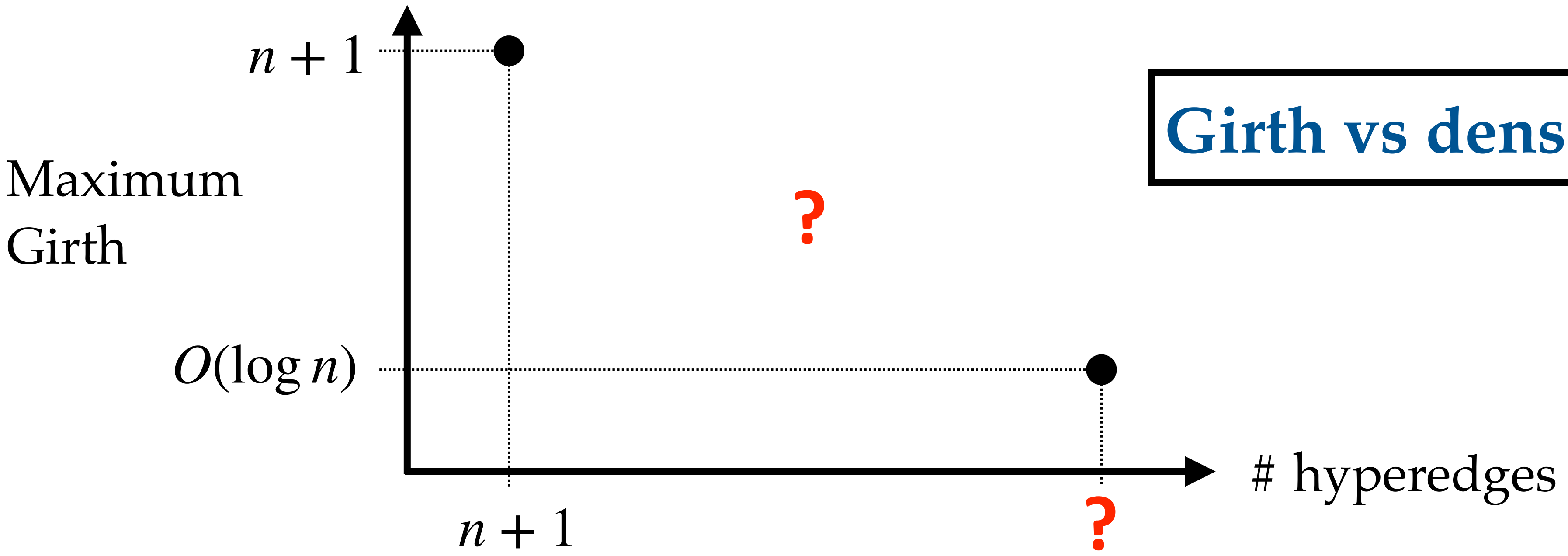

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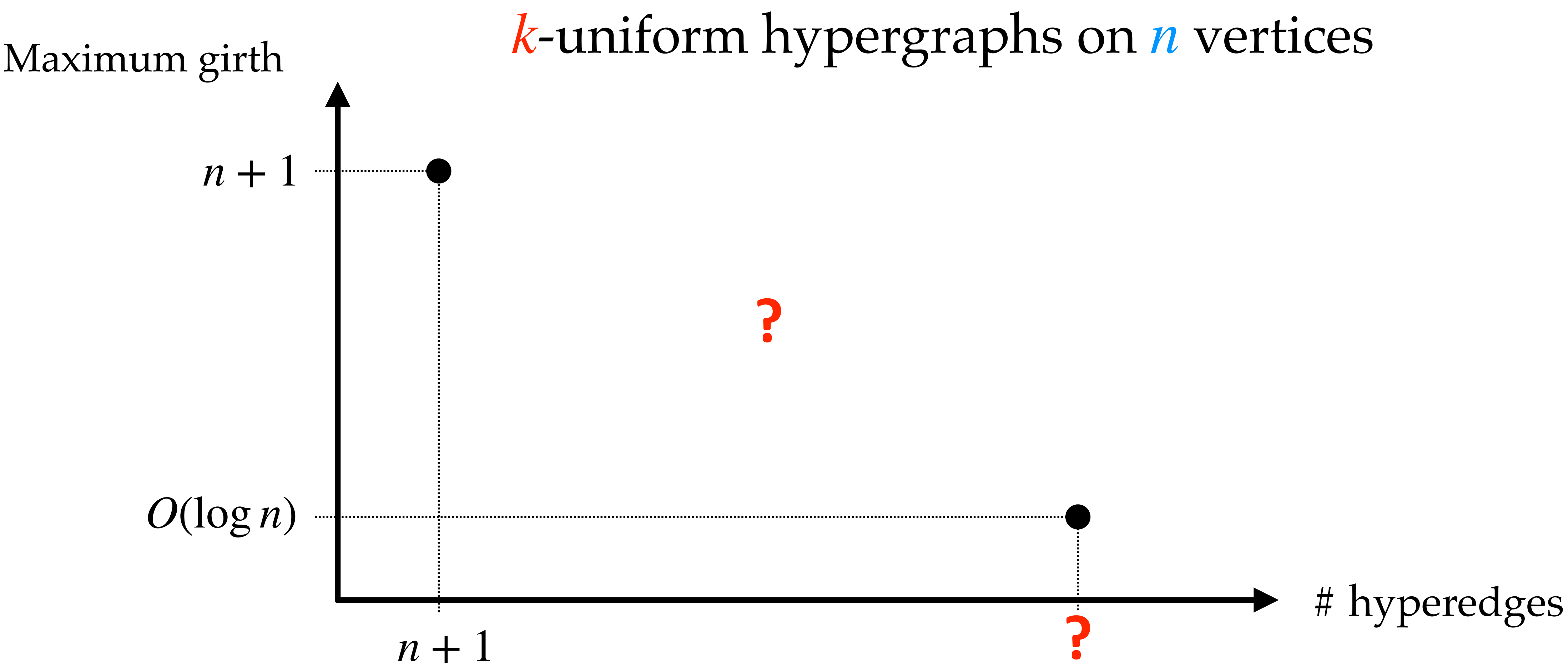
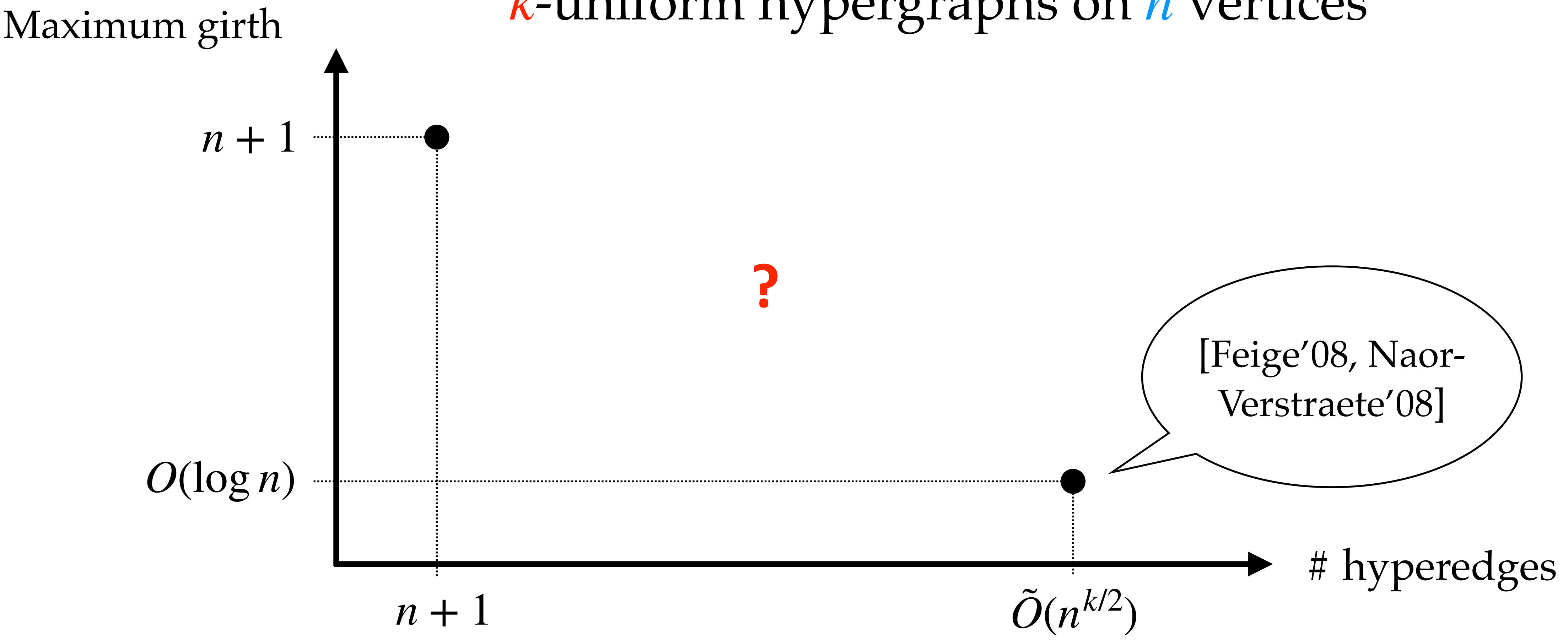


Hypergraph Moore bound

k-uniform hypergraphs on n vertices



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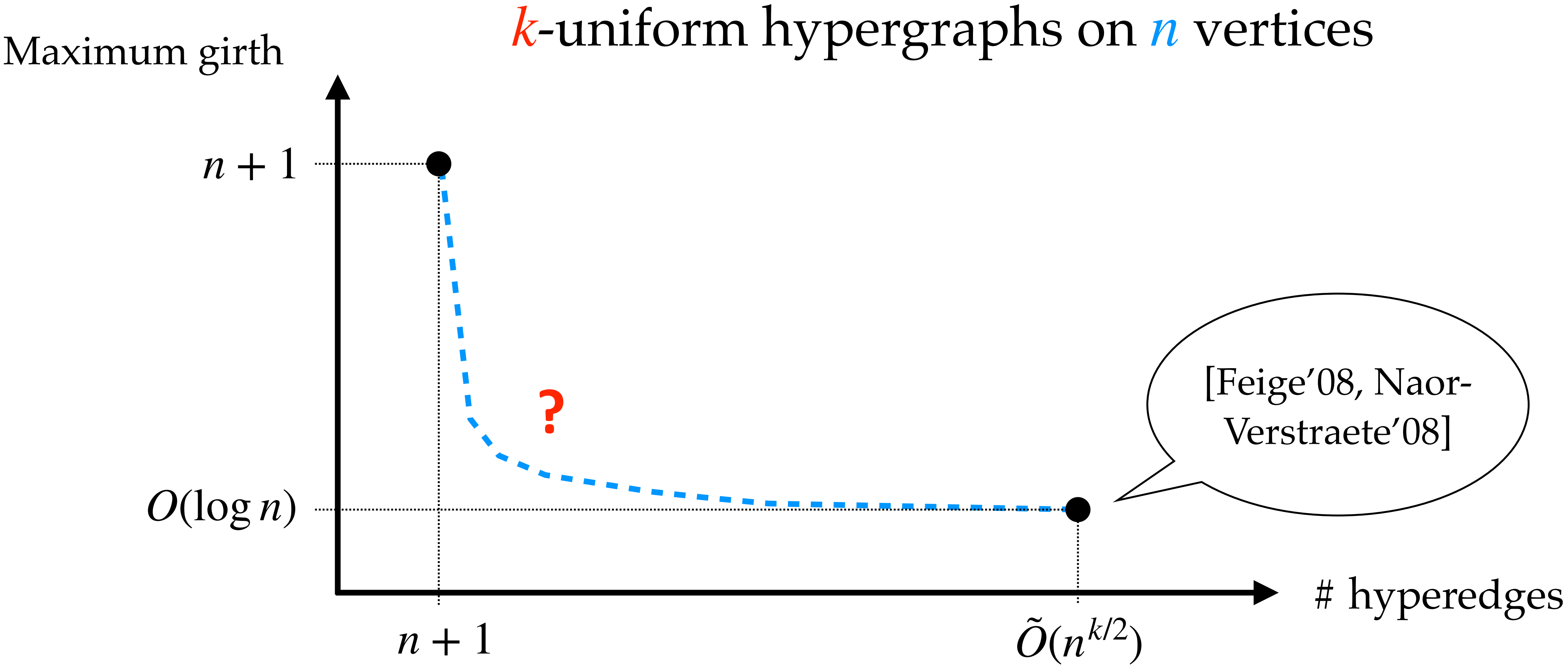

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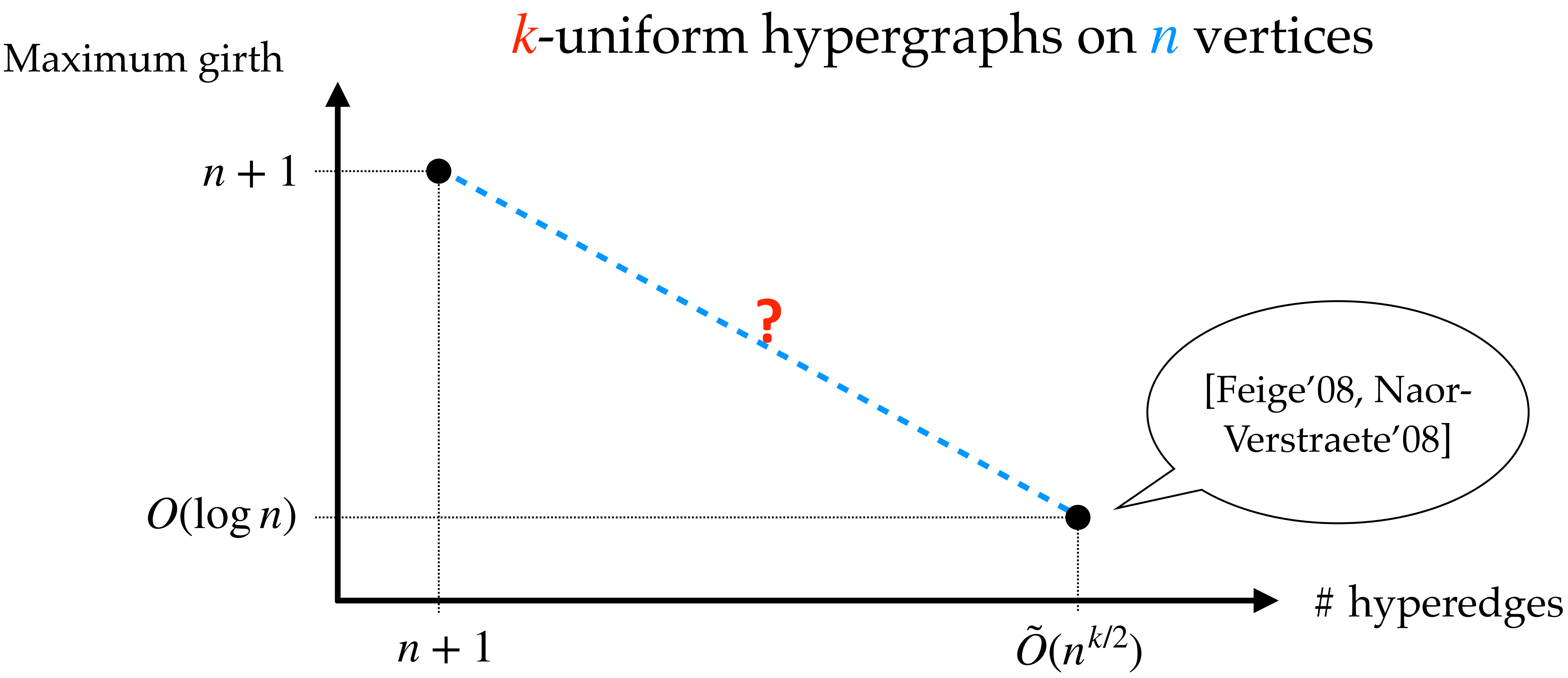
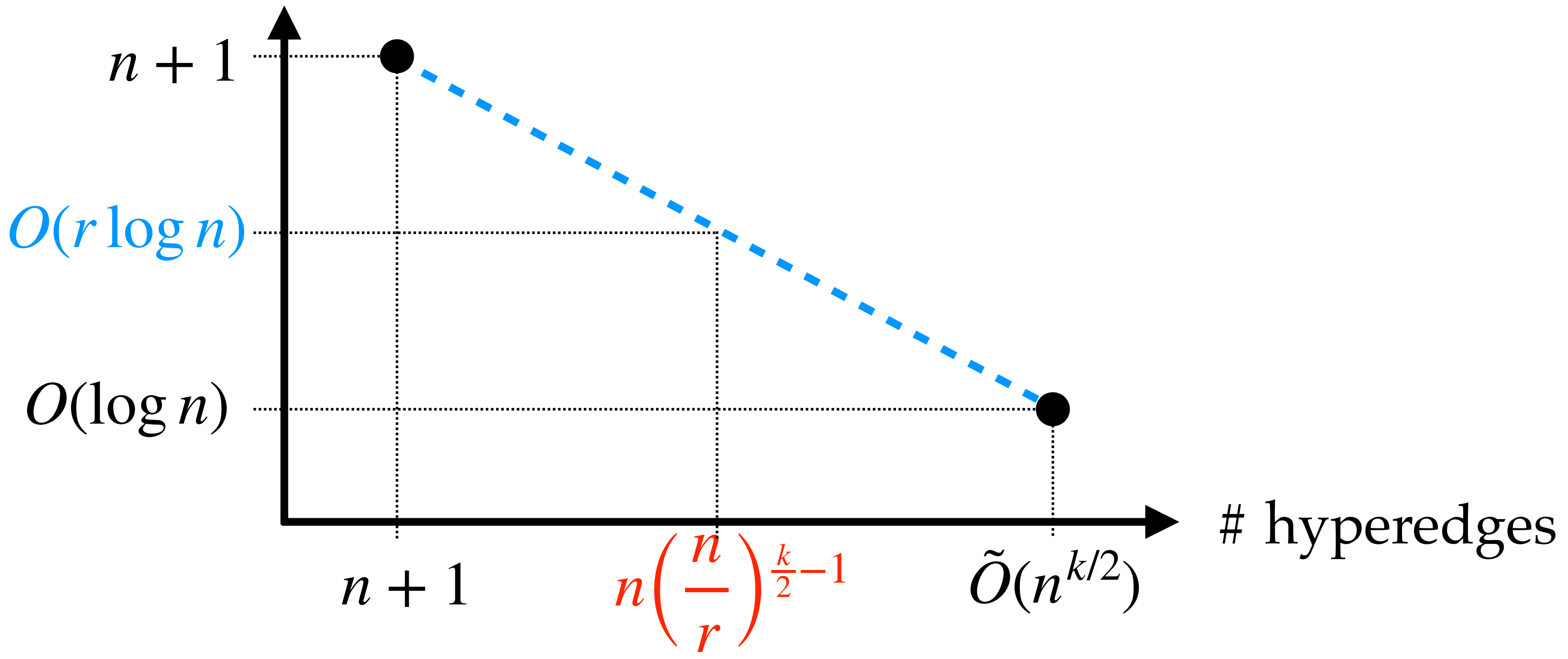


Feige's conjecture

Conjecture: For $1 \leq r \leq n$, every k-uniform hypergraph with n vertices and $m \gtrsim n\left(\frac{n}{r}\right)^{\frac{k}{2}-1}$ hyperedges has an even cover of size $O(r \log n)$.



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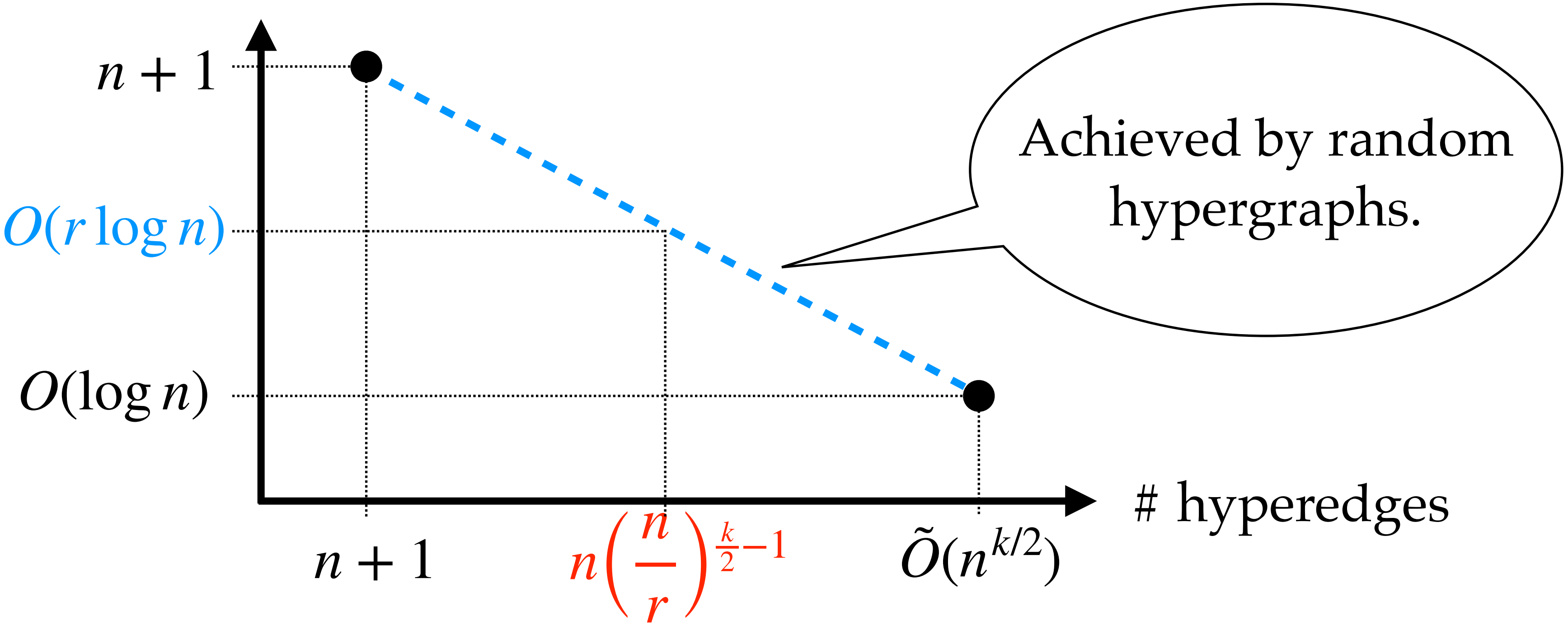

Guruswami, Kothari and Manohar [2022]

Theorem: every k-uniform H with n vertices, $m \gtrsim n\left(\frac{n}{r}\right)^{\frac{k}{2}-1}\log^{4k+1} n$ hyperedges $\Longrightarrow$ even cover of size $O(r \log n)$.

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2 pages for even k!

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[GKM22] loses an extra $\log^3 n$ factor.

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- Last log: likely not real but difficult to remove.

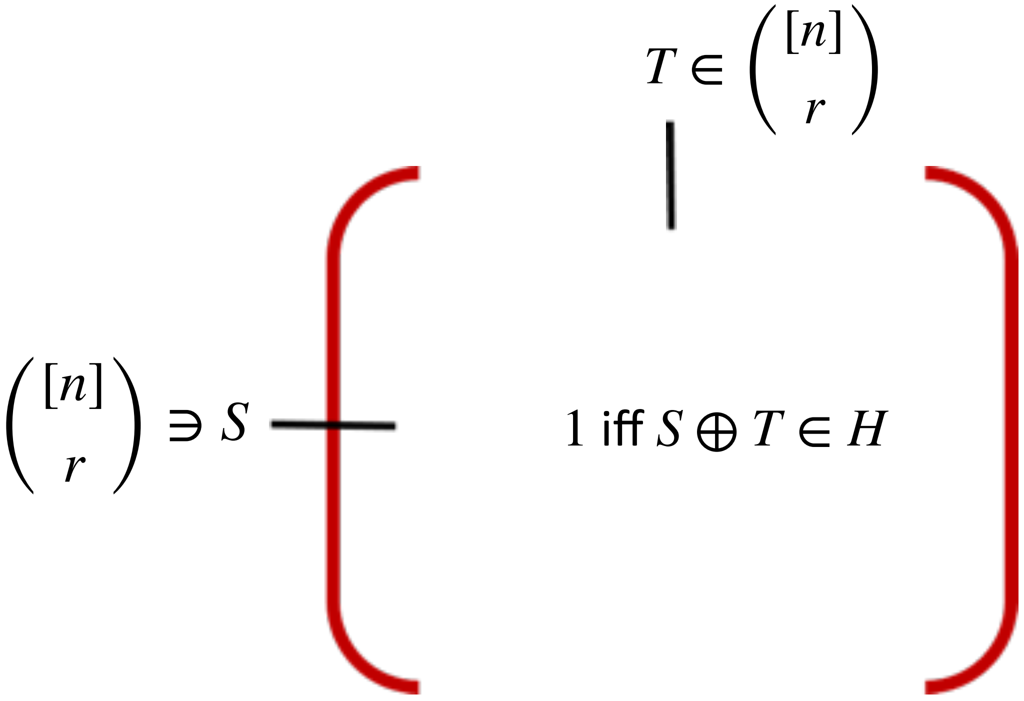
Kikuchi graph

Introduced by [Wein-Alaoui-Moore'19]

Kikuchi graph

Definition. Given parameter r, the Kikuchi graph (associated to the hypergraph H) is a graph on vertex set $\binom{[n]}{r}$ and two vertices S, T are connected if $S \oplus T \in H$.

Symmetric difference



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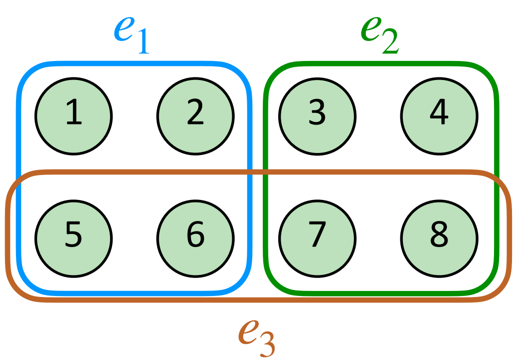
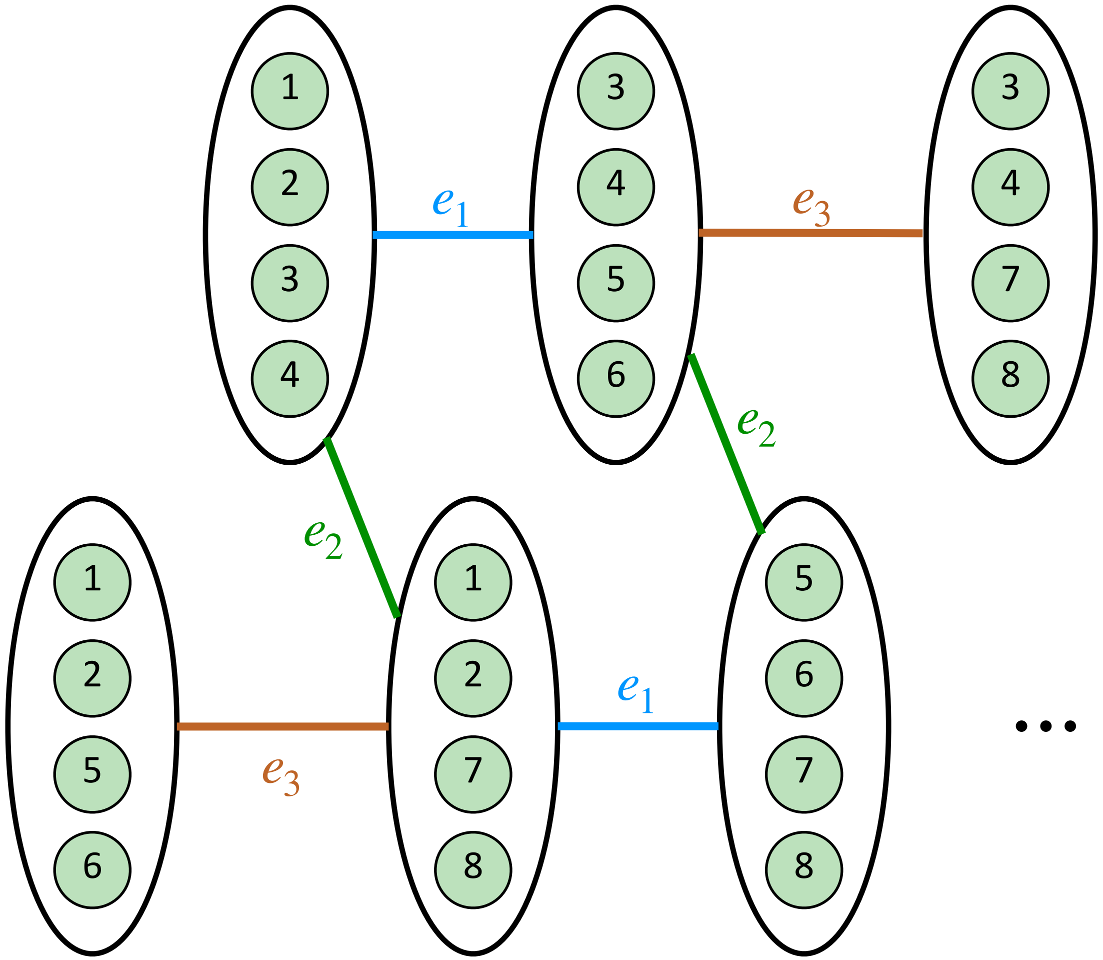


4-uniform hypergraph H



Kikuchi graph with $r = 4$

Cycles in Kikuchi $\leftrightarrow$ even covers

Claim: Cycles in Kikuchi graph $\implies$ even covers in H.*

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Cycle: $S_1 \xrightarrow{C_1} S_2 \xrightarrow{C_2} S_3 \cdots \longrightarrow S_\ell \xrightarrow{C_\ell} S_1$.

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$$S_1 \oplus S_2 = C_1$$
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$$\vdots$$
$$S_\ell \oplus S_1 = C_\ell$$

$$\emptyset = C_1 \oplus \cdots \oplus C_\ell$$

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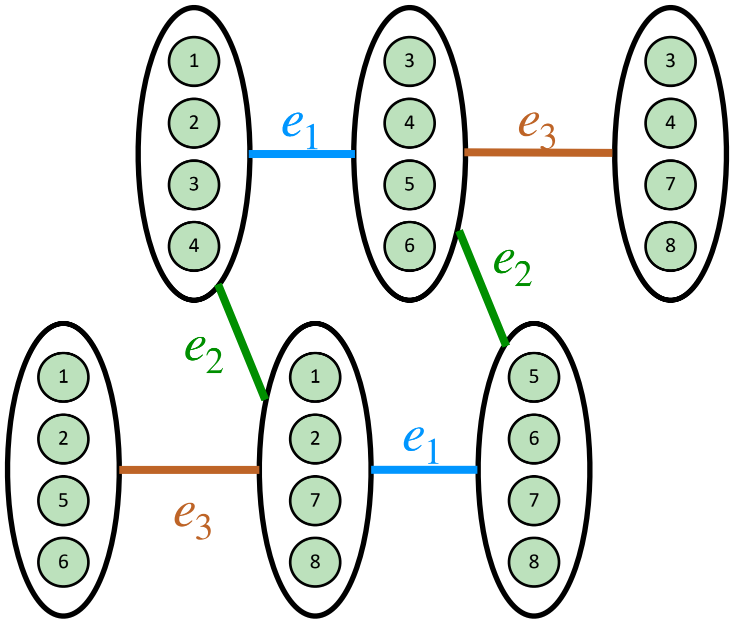

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- **Trivial** cycles: each hyperedge appears even number of times.
- **Non-trivial** cycles: **even covers**!

Proof: cleverly count these cycles!

Open questions

Tight bounds for Feige's conjecture?

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