Ellipsoid Fitting Up to a Constant

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Overview

1. **Ellipsoid Fitting Conjecture**
2. Constructing an Ellipsoid
3. Analysis via Graph Matrices
4. A Local Machinery for Tight Norm Bounds

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- Geometrically, $\Lambda \in R^{d \times d}$ is an **ellipsoid** centered at origin that **passes through all m points on boundary.**

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Sharp transition!

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(Negative side via dimension-argument)

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    —> w.h.p., the linear system cannot be satisfied.
- **Open Question**: Can we improve upon this bound? PSDness?

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Is the polylog dependence tight?

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- **Constant-factor** away from the conjecture

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3. Graph Matrices
4. Tight Norm Bounds for Graph Matrices

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    - W.h.p., $|v_i^T \Lambda v_i - 1| \leq \frac{\log d}{\sqrt{d}}$.

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How do we pick w?

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$$= \|v_i\|_2^2 - \underbrace{(\|v_i\|_2^2 - 1)}_{\eta_i}$$

If we pick w s.t. $Mw = \eta$

$= 1$ (**Exactly** satisfying the constraint)

Remaining questions

Picking $w \in \mathbf{R}^m$ s.t. $Mw = \eta$, and set $\Lambda = Id_d - \sum_{a=1}^{m} w_a v_a v_a^T$

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Construction from [KD '22]

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Both boil down to studying spectral norm of random matrices with polynomial entries

Why norm bounds?

(Vector w is well-defined)

$M \in \mathbf{R}^{m \times m} : M[i,j] = \langle v_i, v_j \rangle^2$

$\eta \in \mathbf{R}^m : \eta_i := \|v_i\|_2^2 - 1$

Why norm bounds?

(Vector w is well-defined)

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Our focus: showing $A = Id + E$ for $\|E\| < 1$

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Overview

1. Ellipsoid Fitting Conjecture
2. Constructing an Ellipsoid
3. **Analysis via Graph Matrices**
4. A Local Machinery for Tight Norm Bounds

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  - eg. Planted Clique, Sparse Independent Set, Densest-k-Subgraph…

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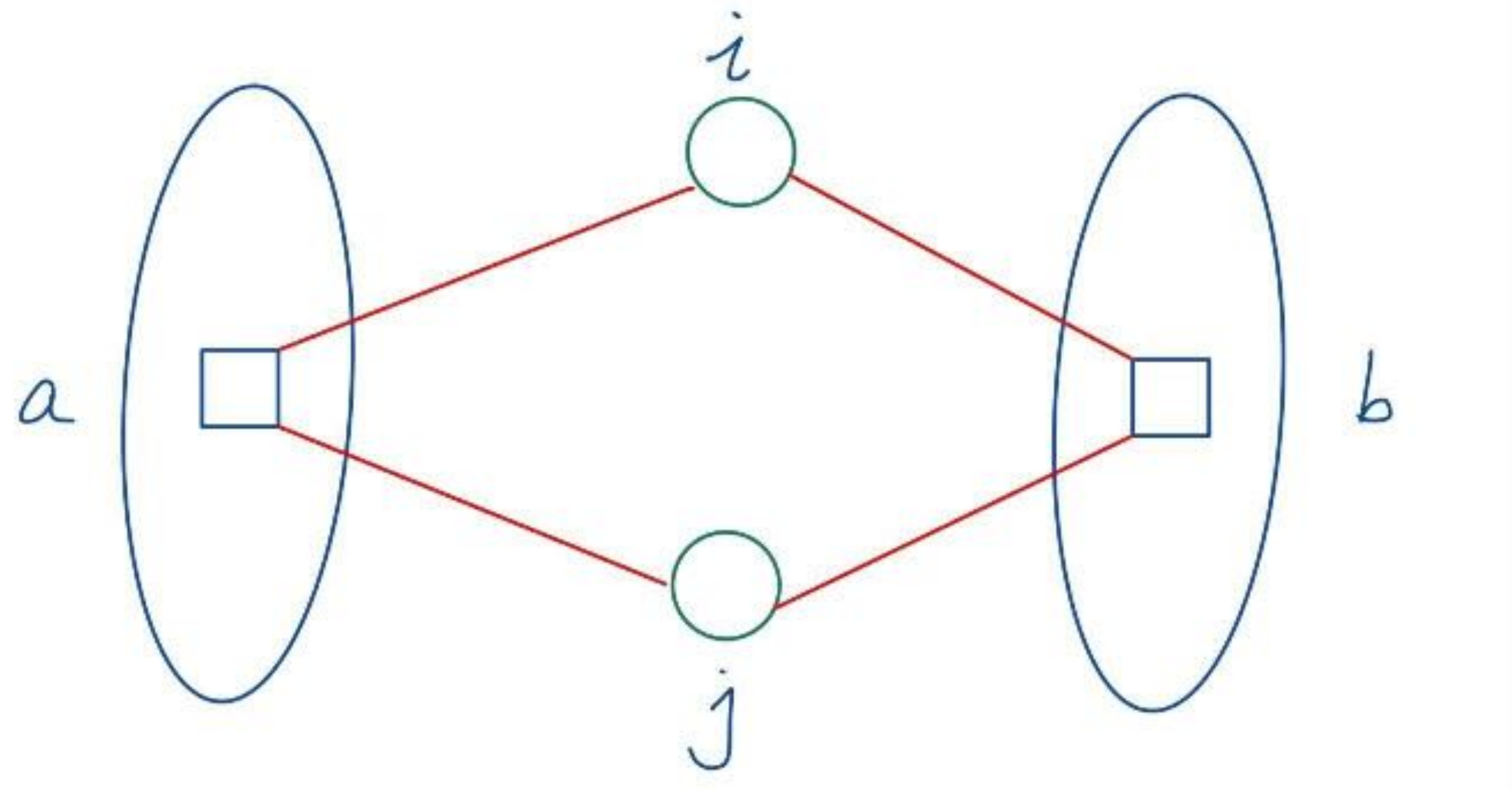

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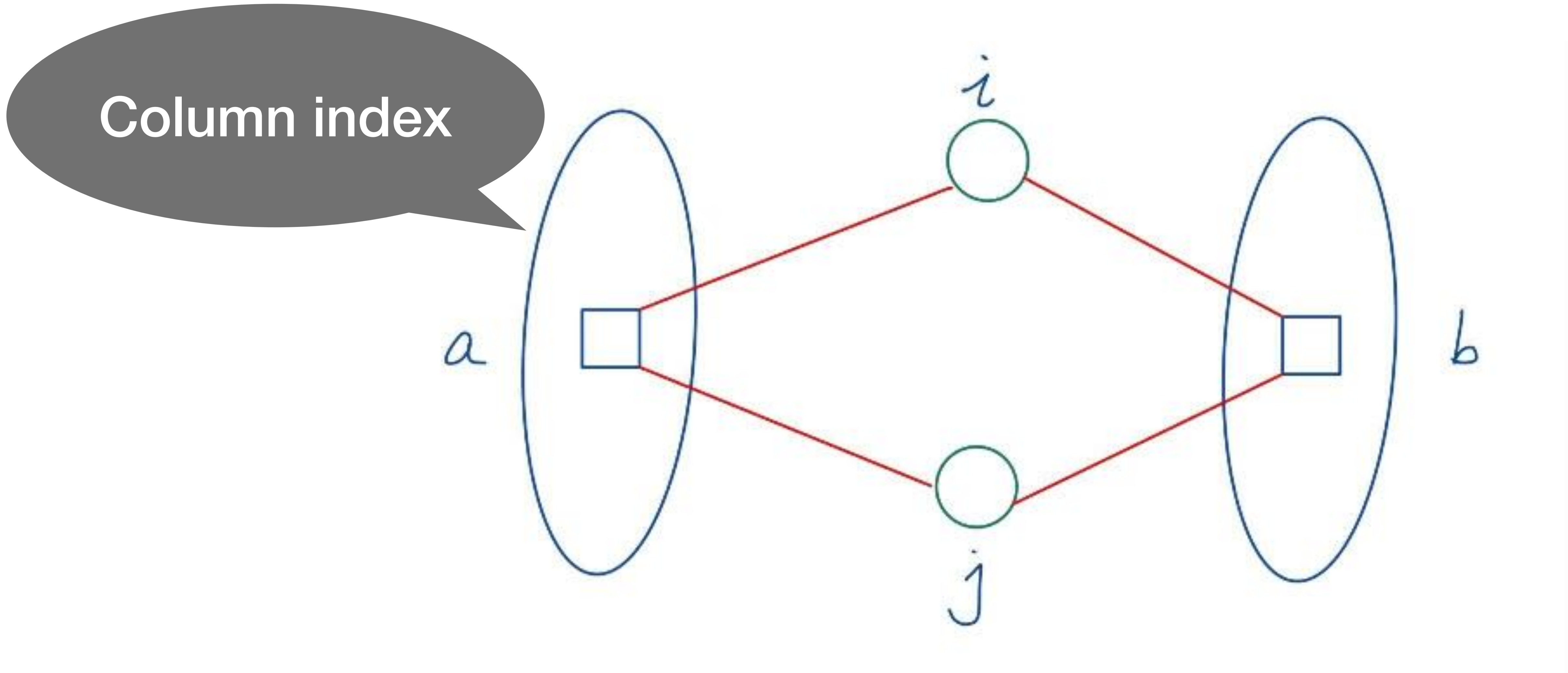

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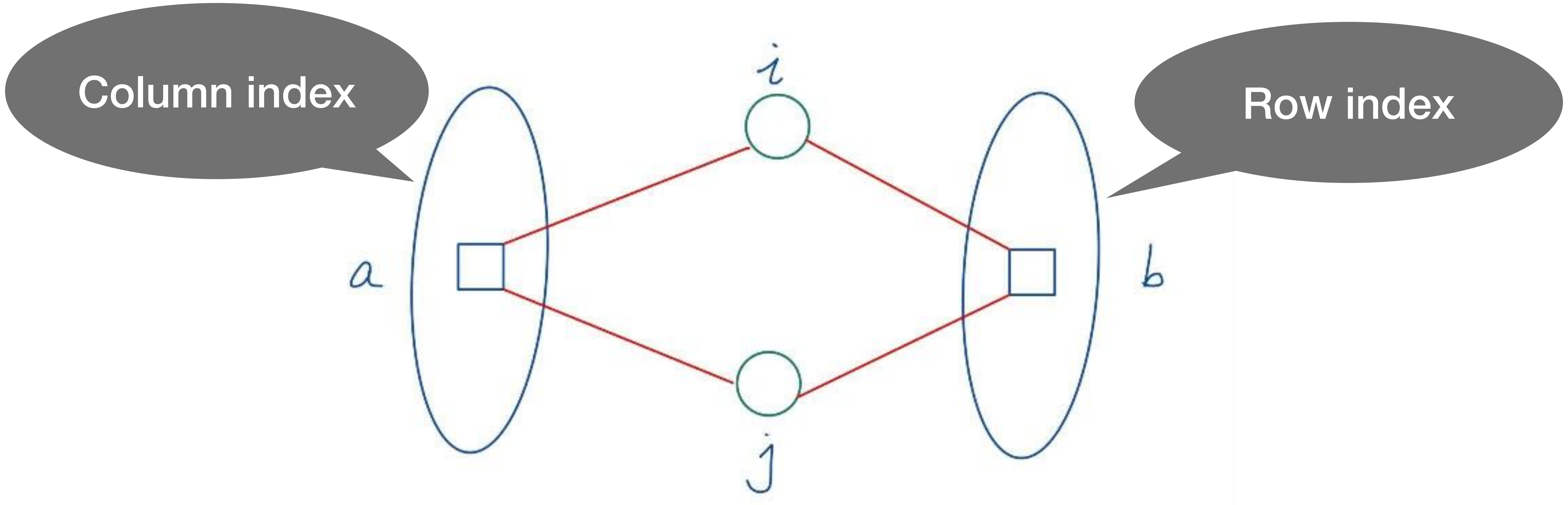

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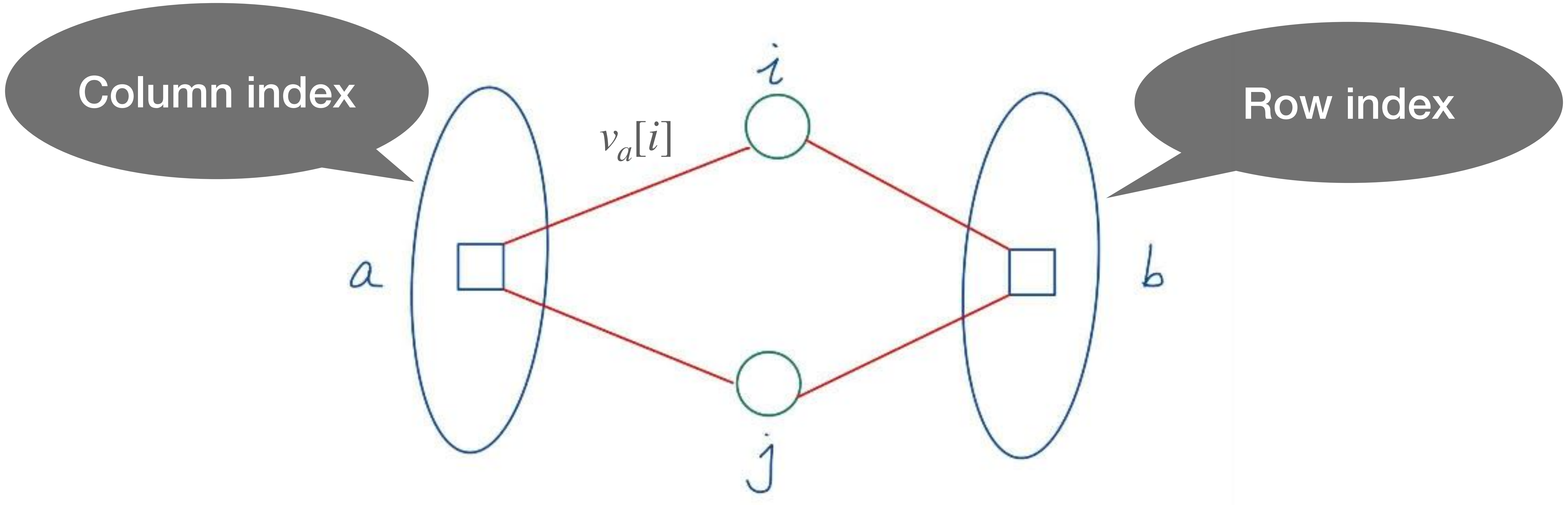

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Overview

1. Ellipsoid Fitting Conjecture
2. Constructing an Ellipsoid
3. Analysis via Graph Matrices
4. **A Local Machinery for Tight Norm Bounds**

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Similar lemmas were known before for G.O.E. and adjacency matrix only

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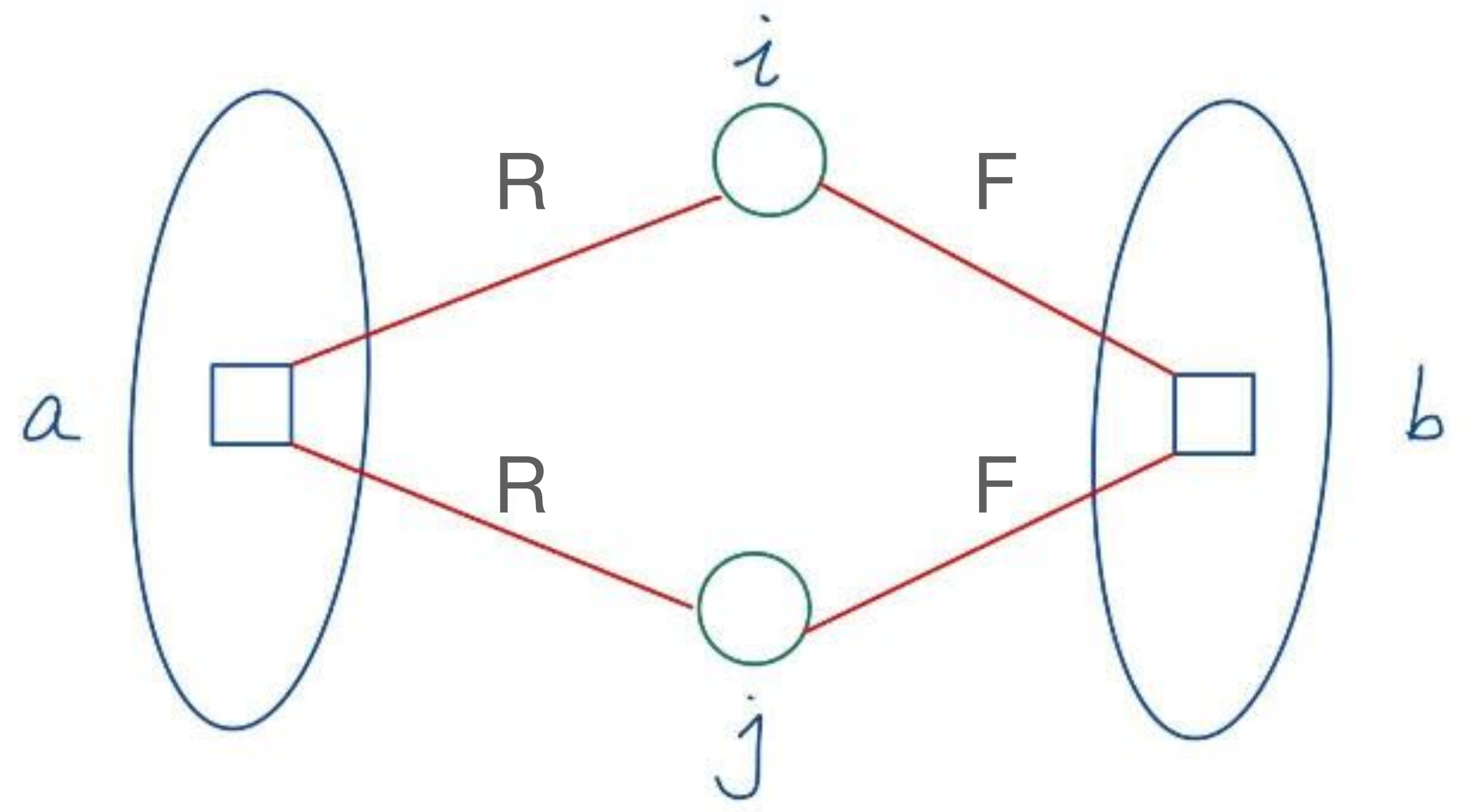

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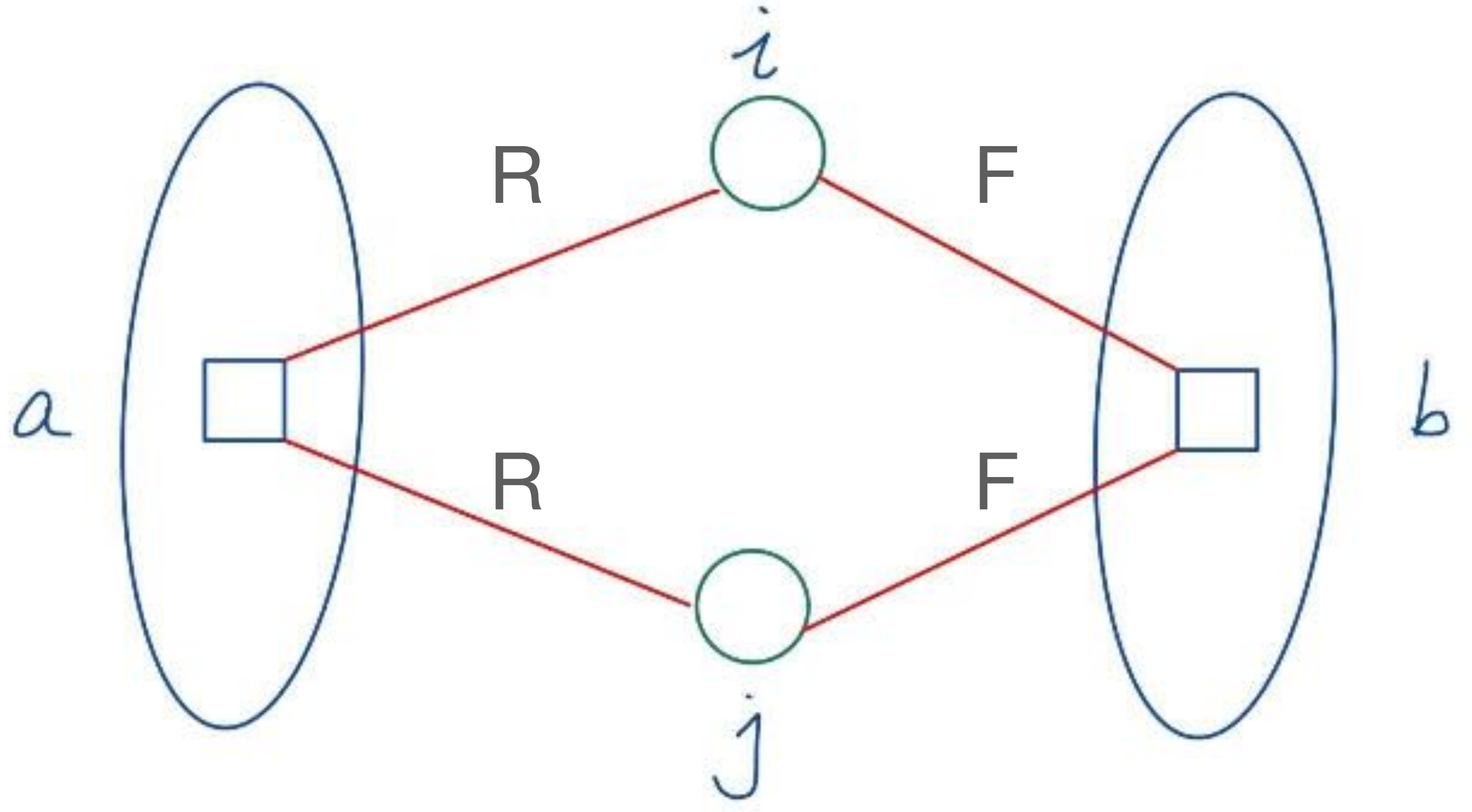

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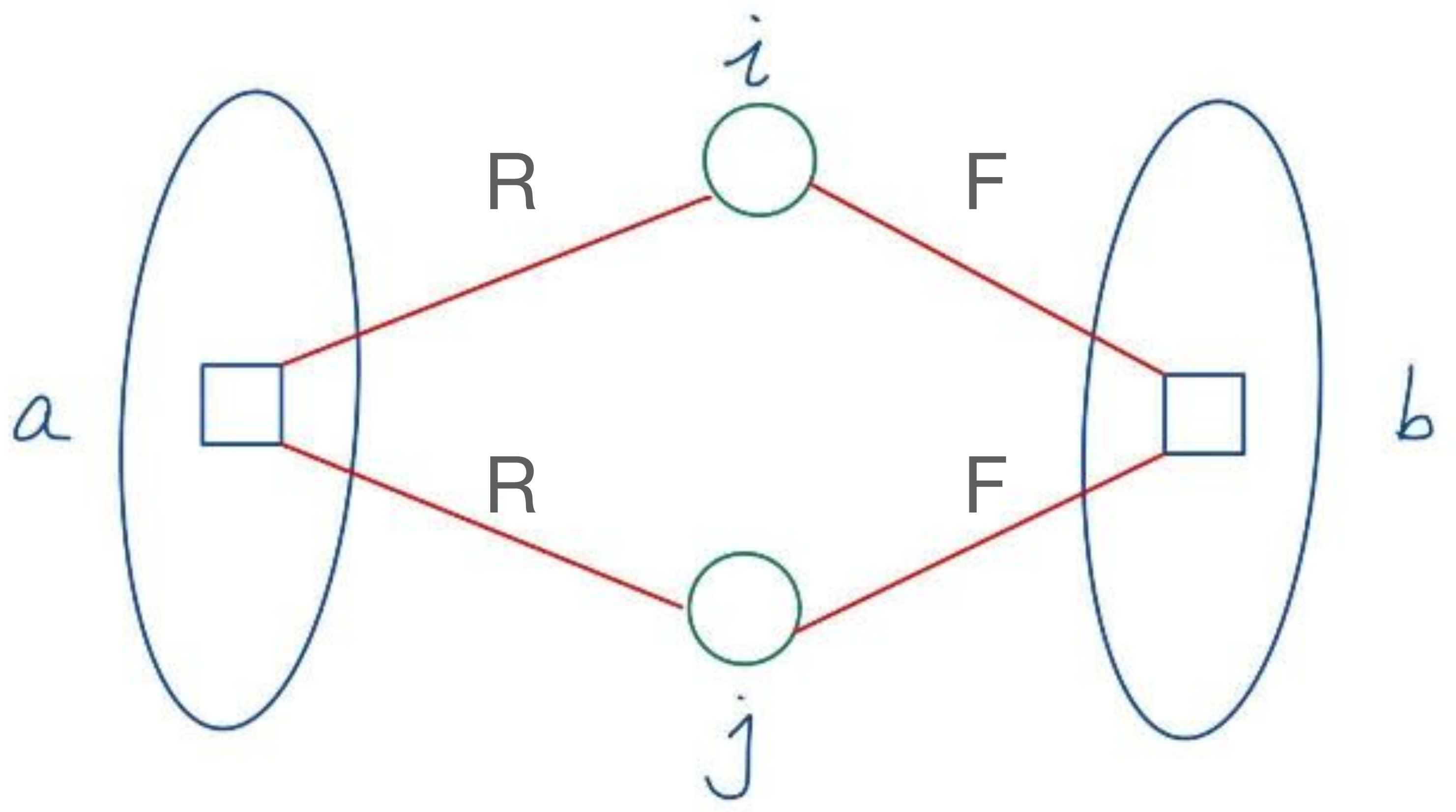

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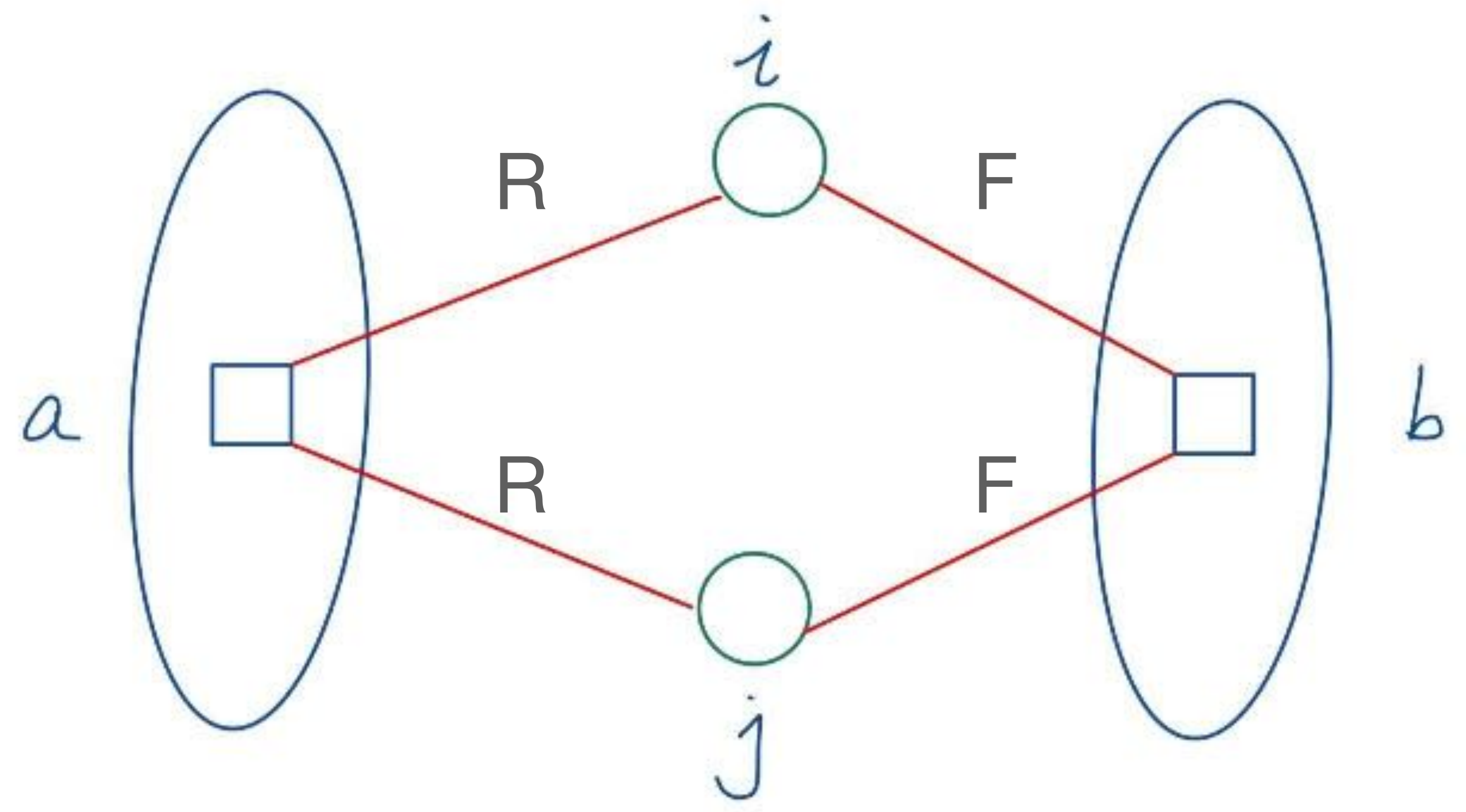

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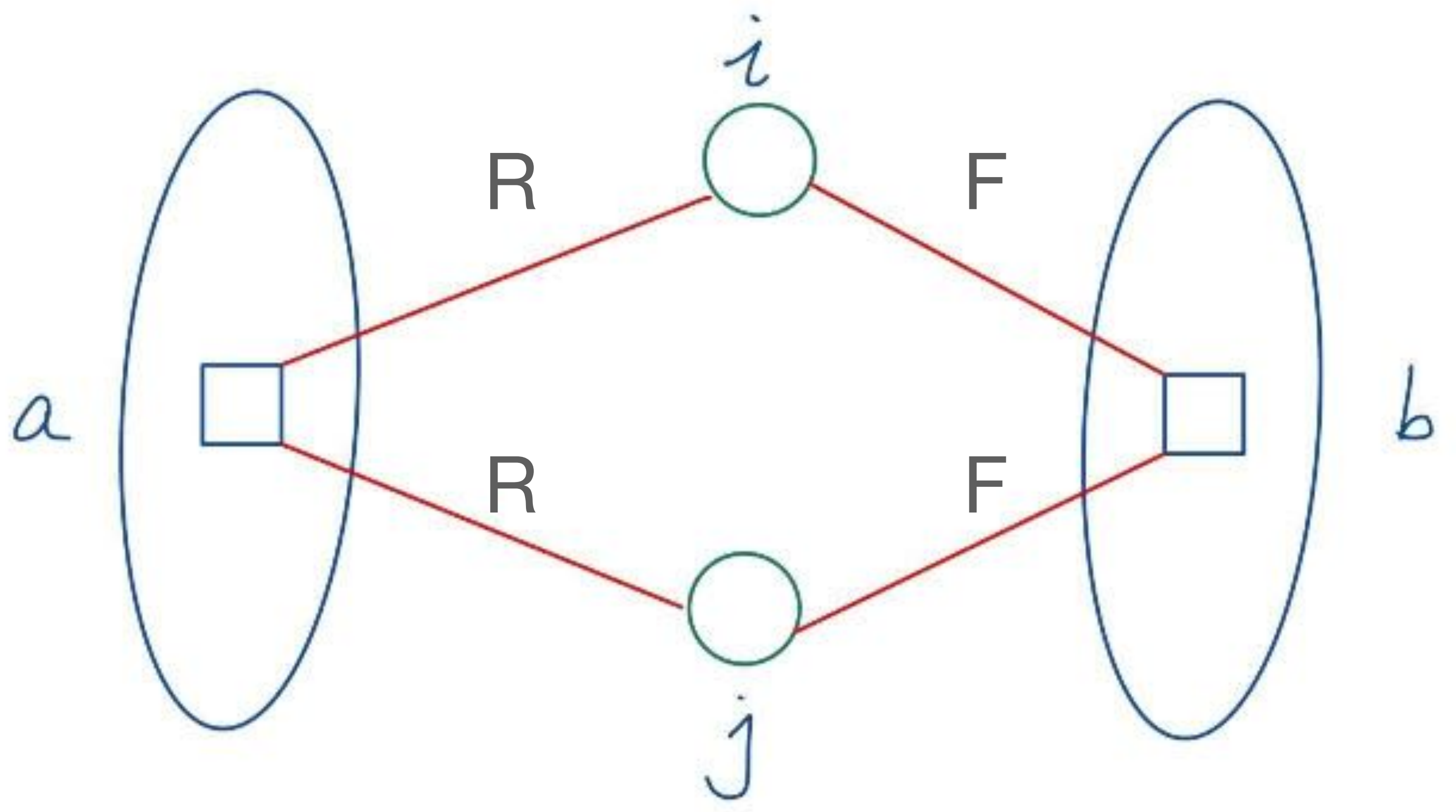

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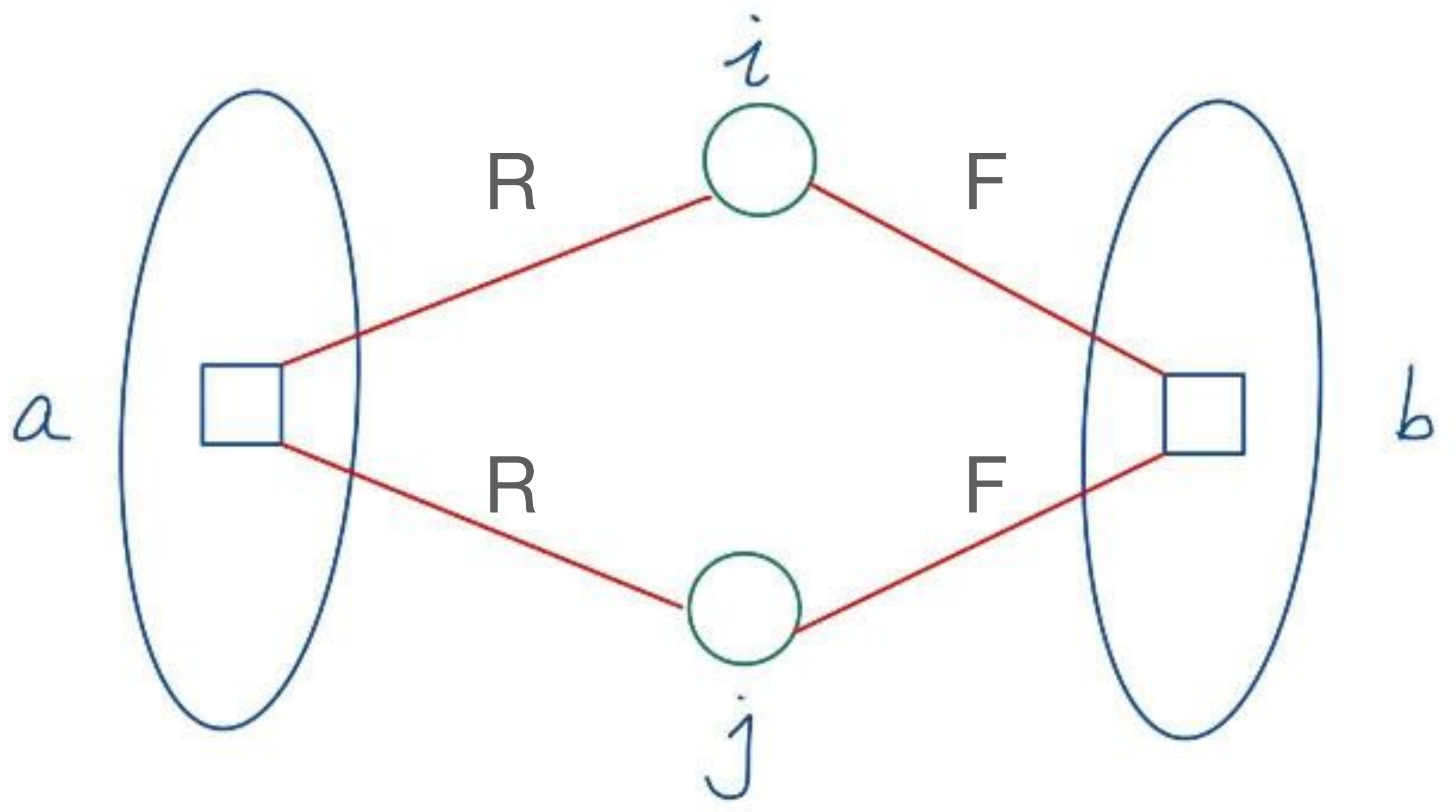

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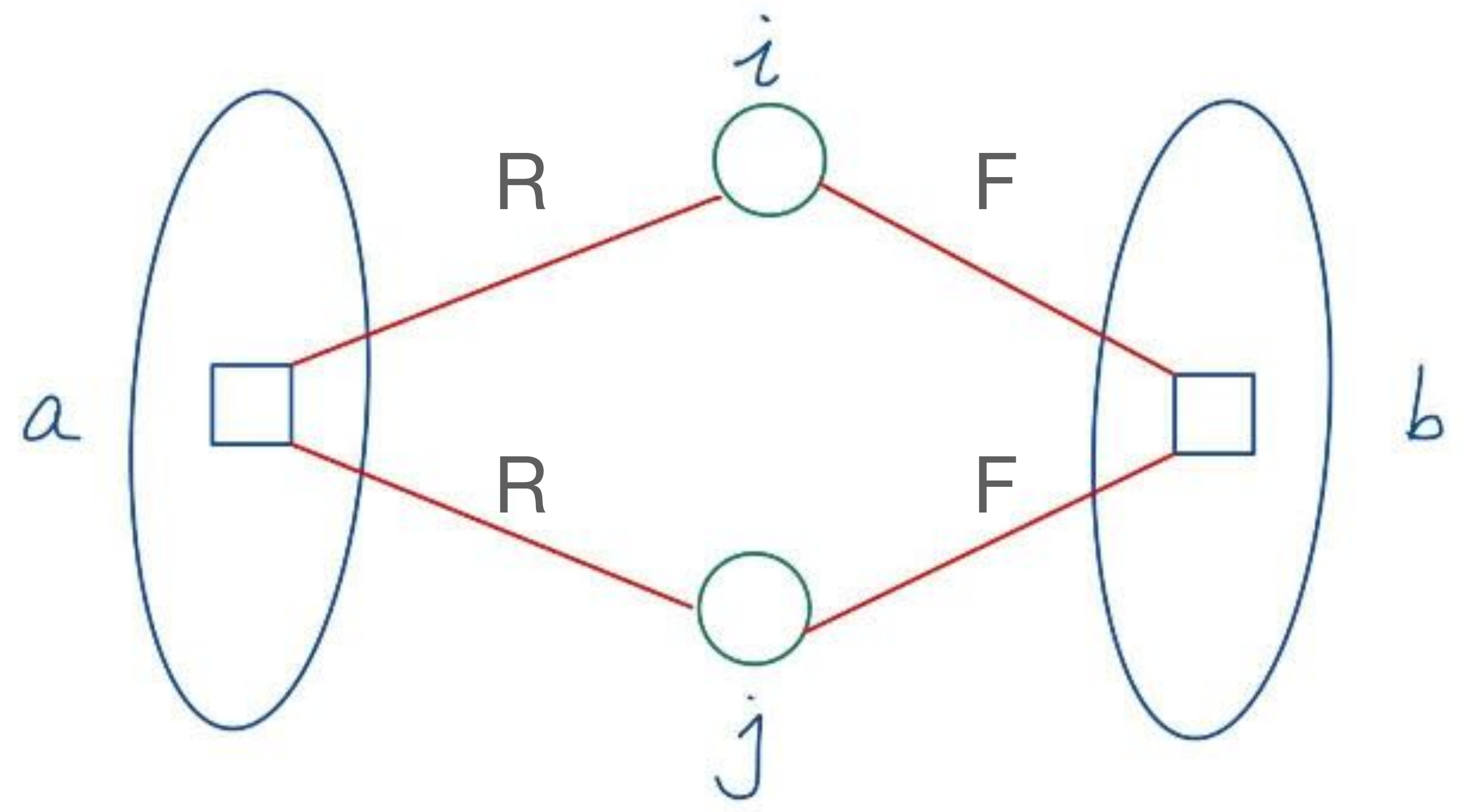

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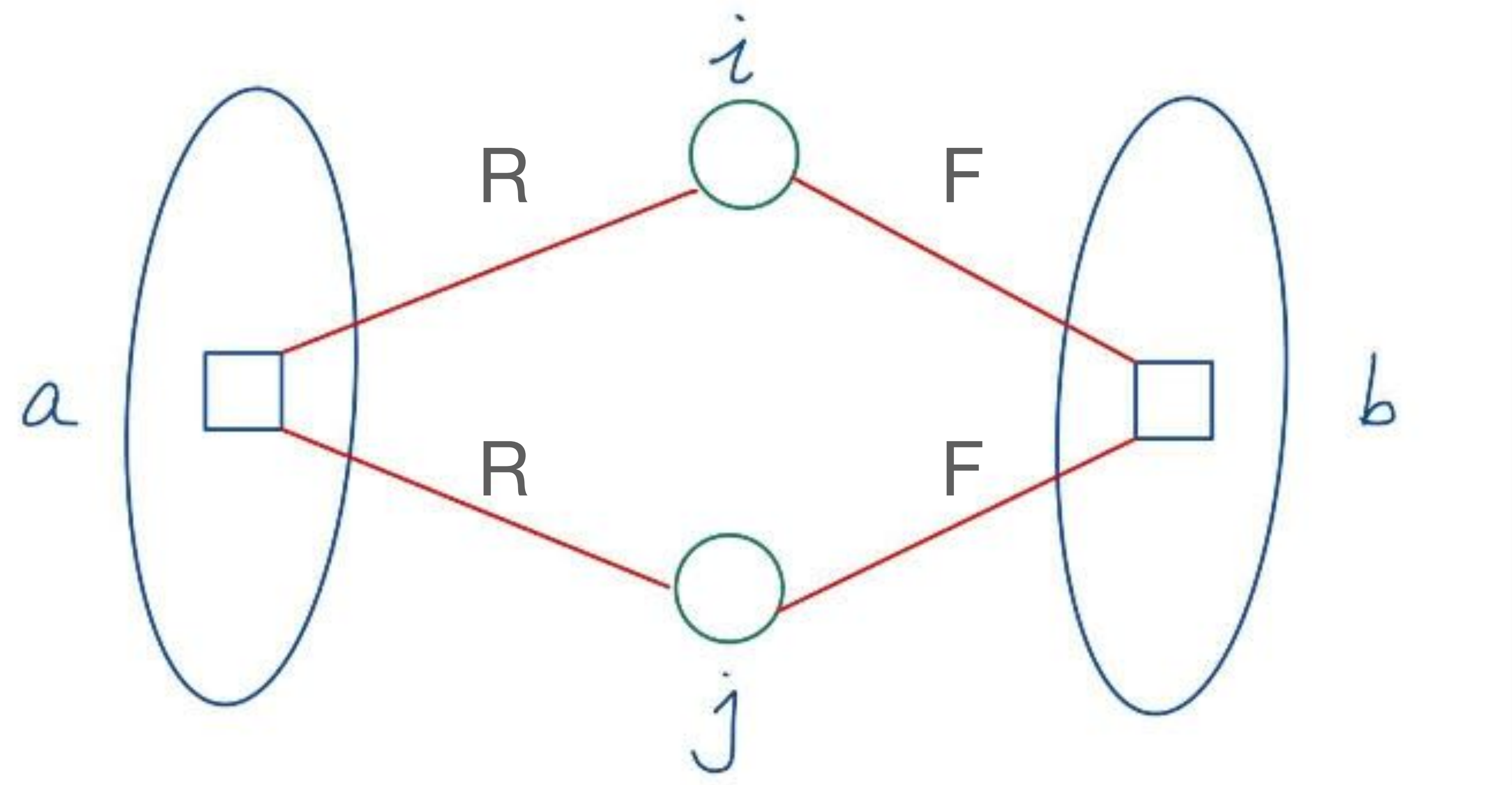

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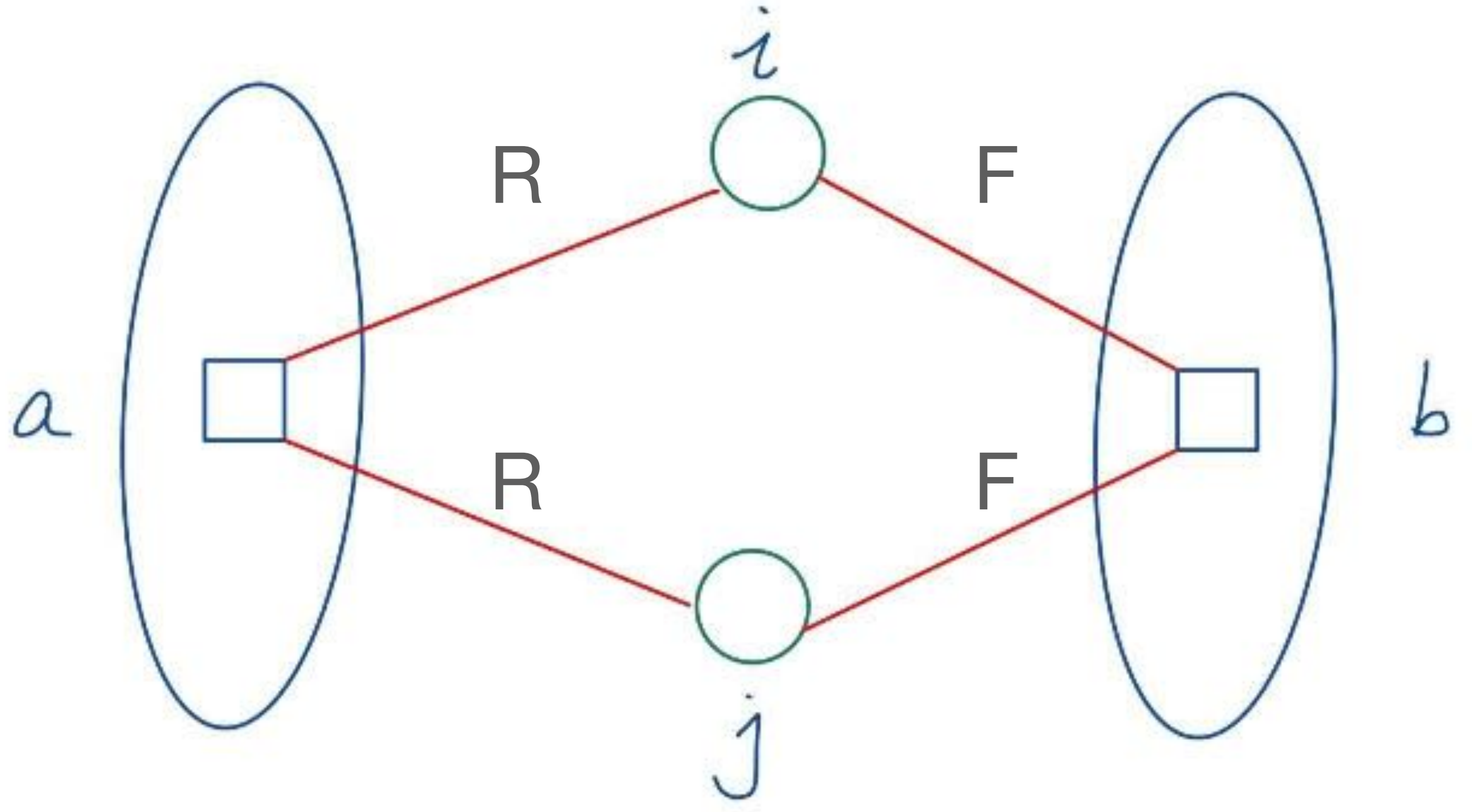

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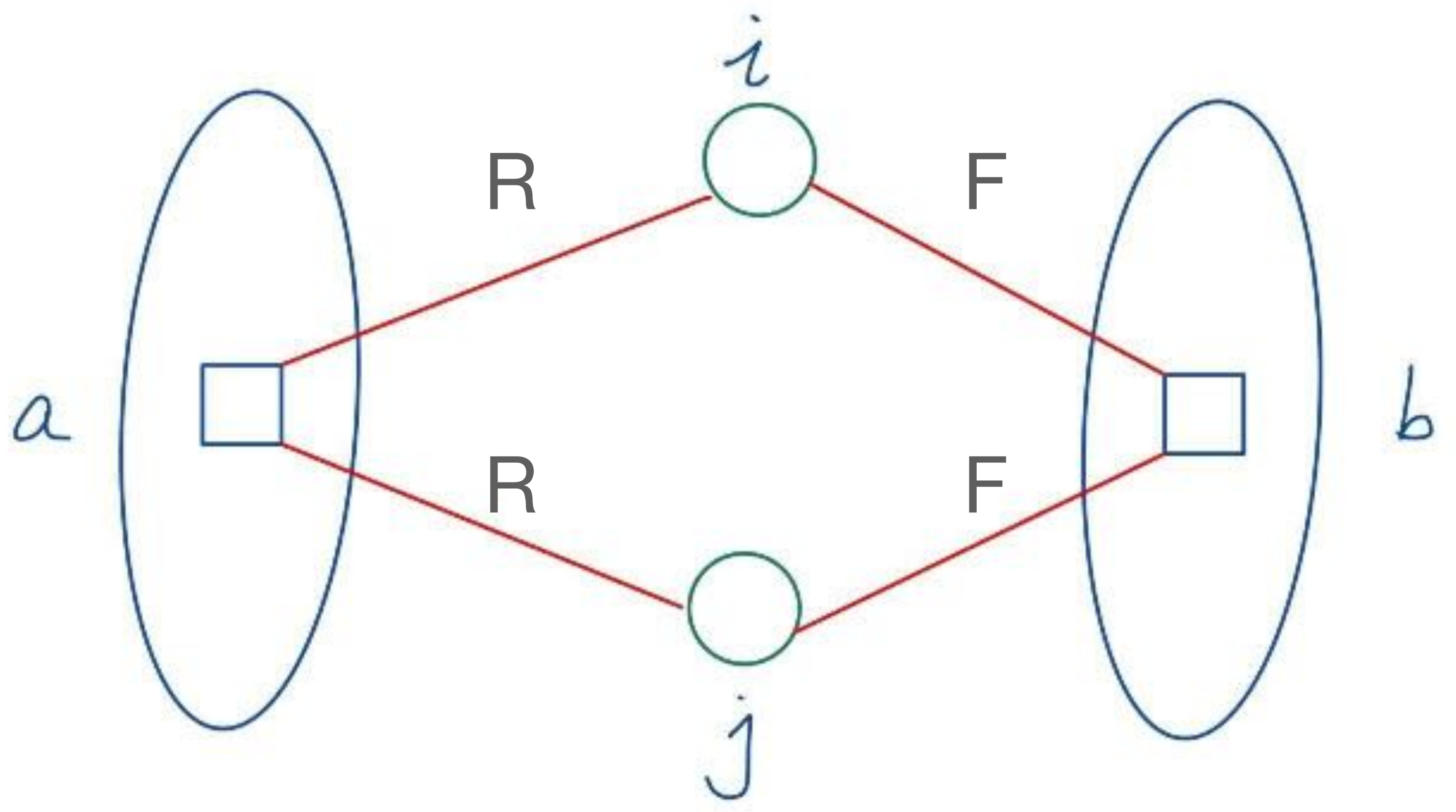

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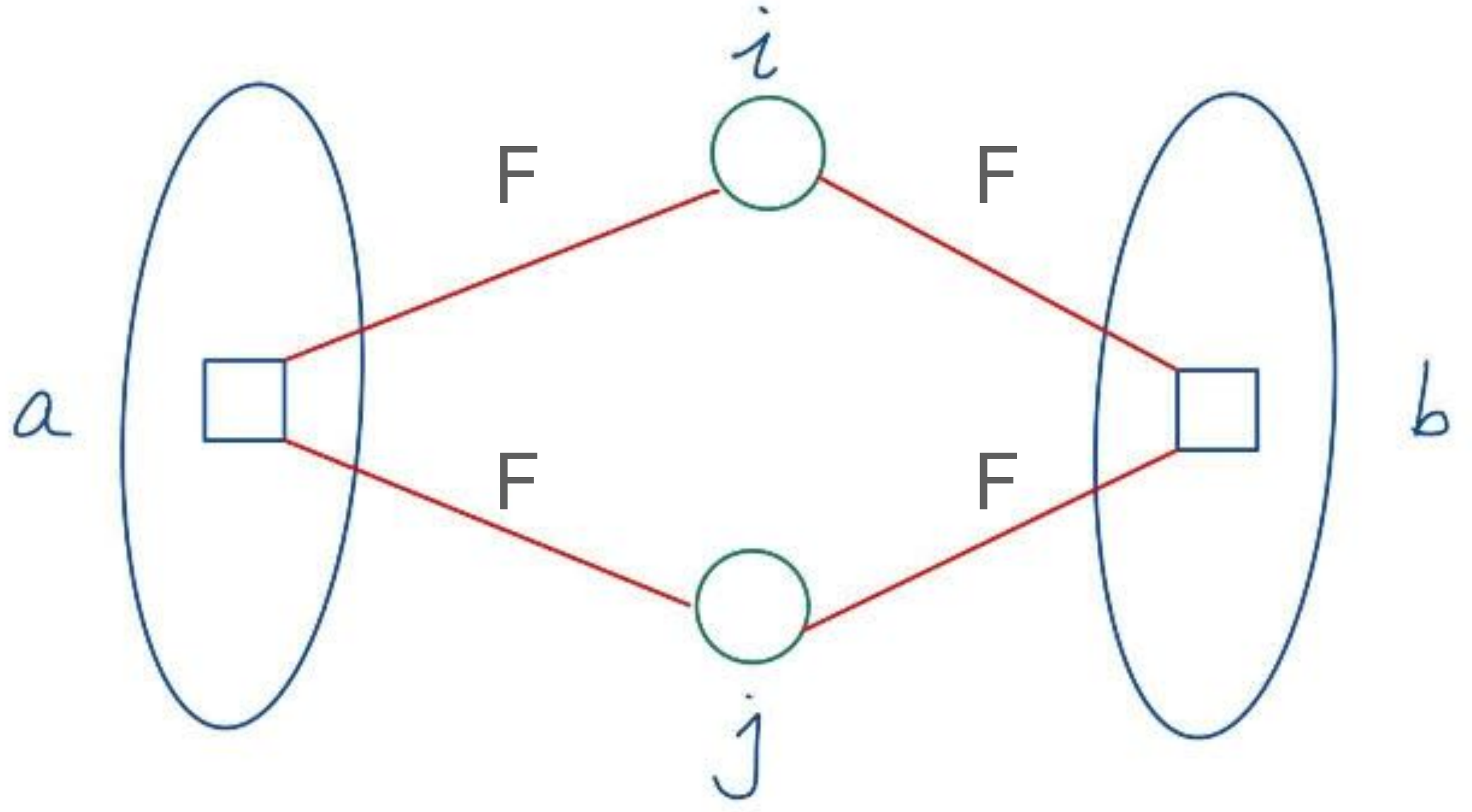

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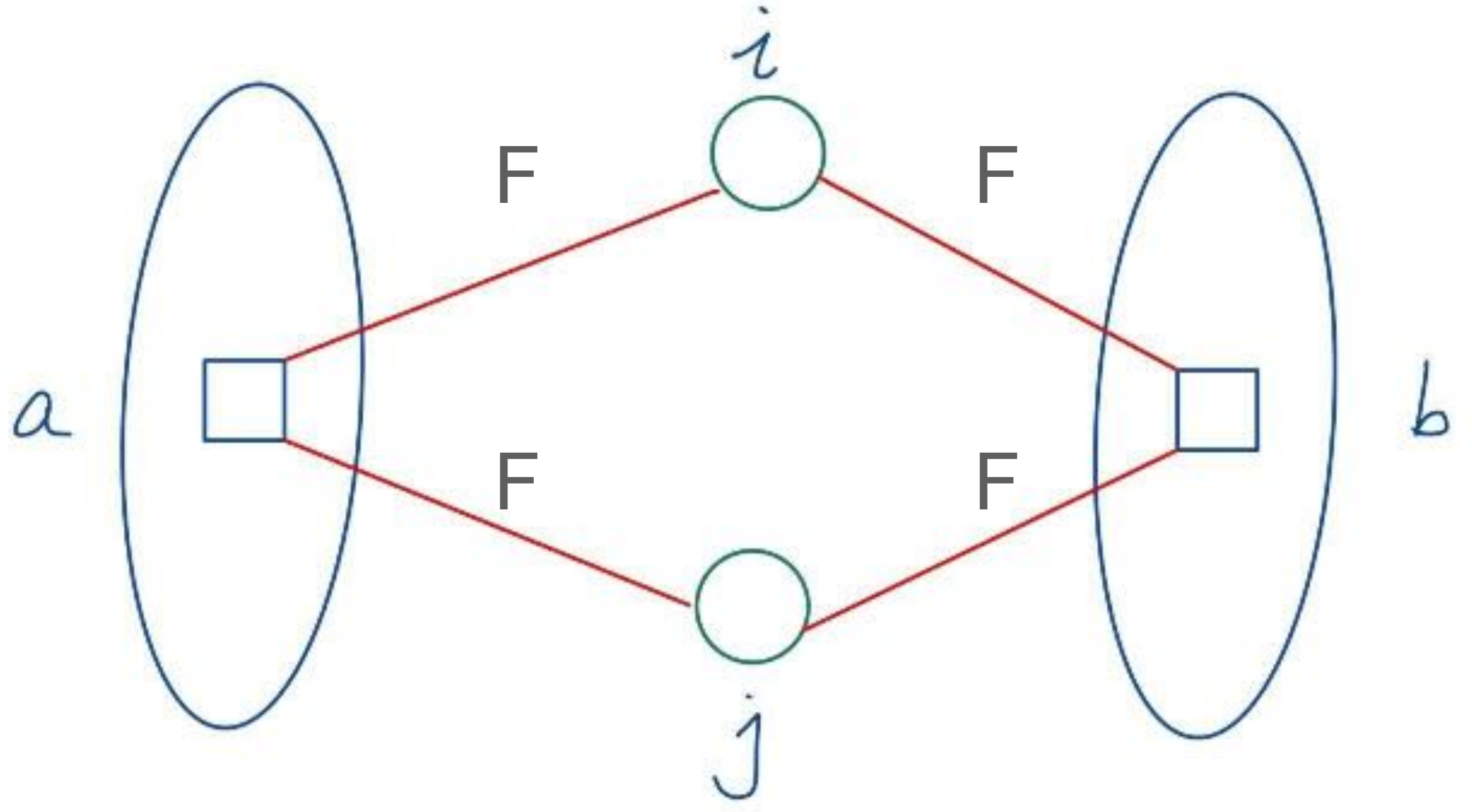

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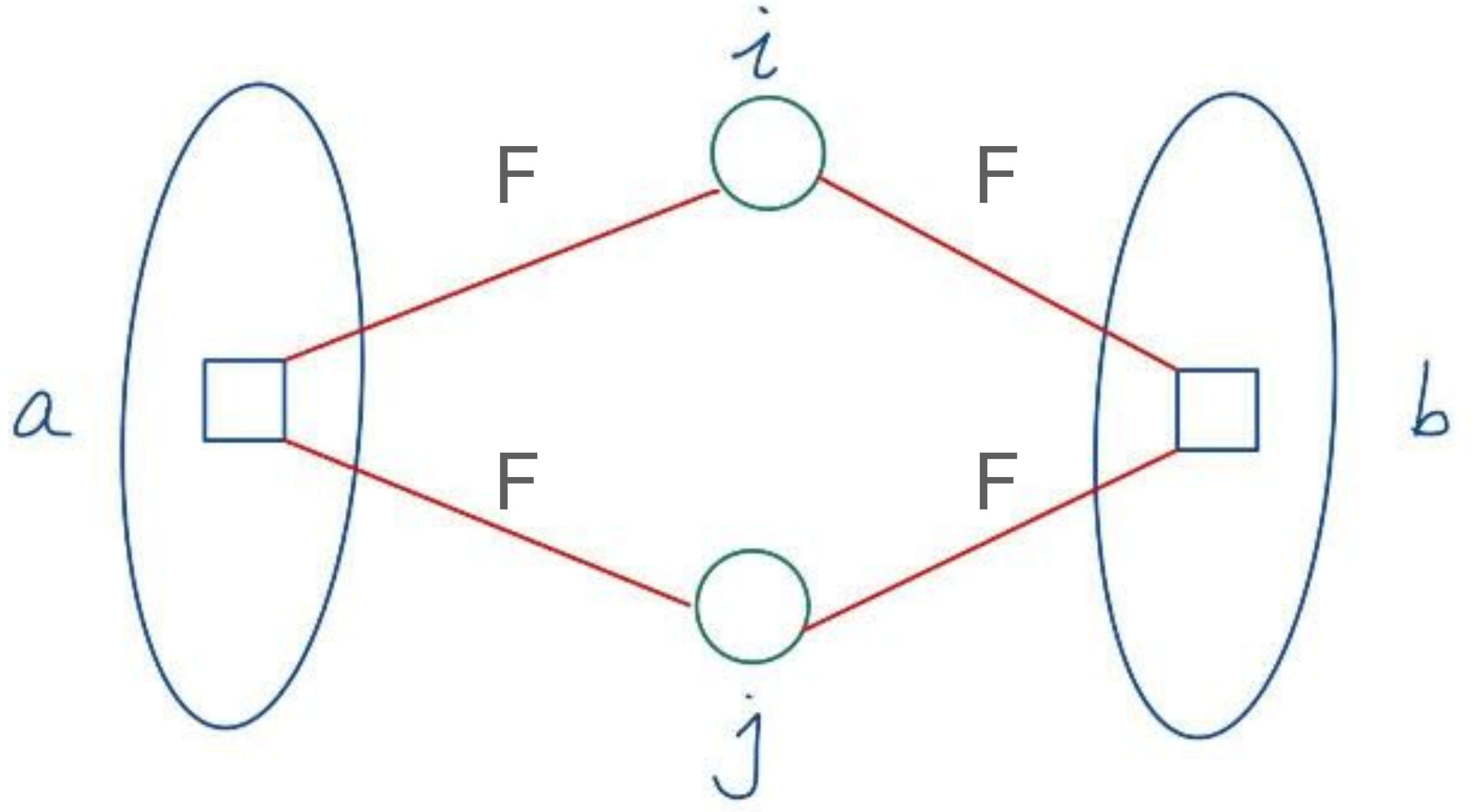

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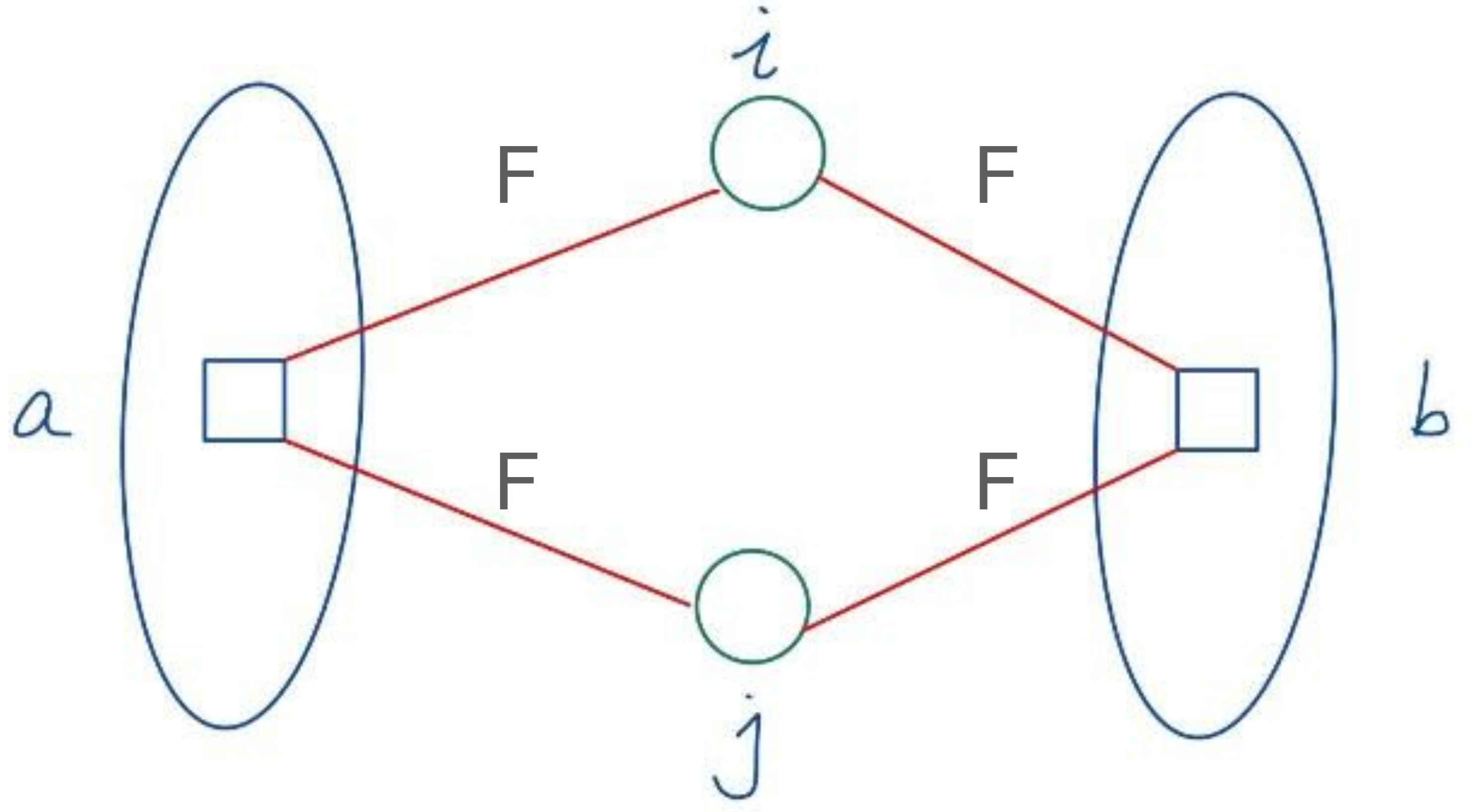

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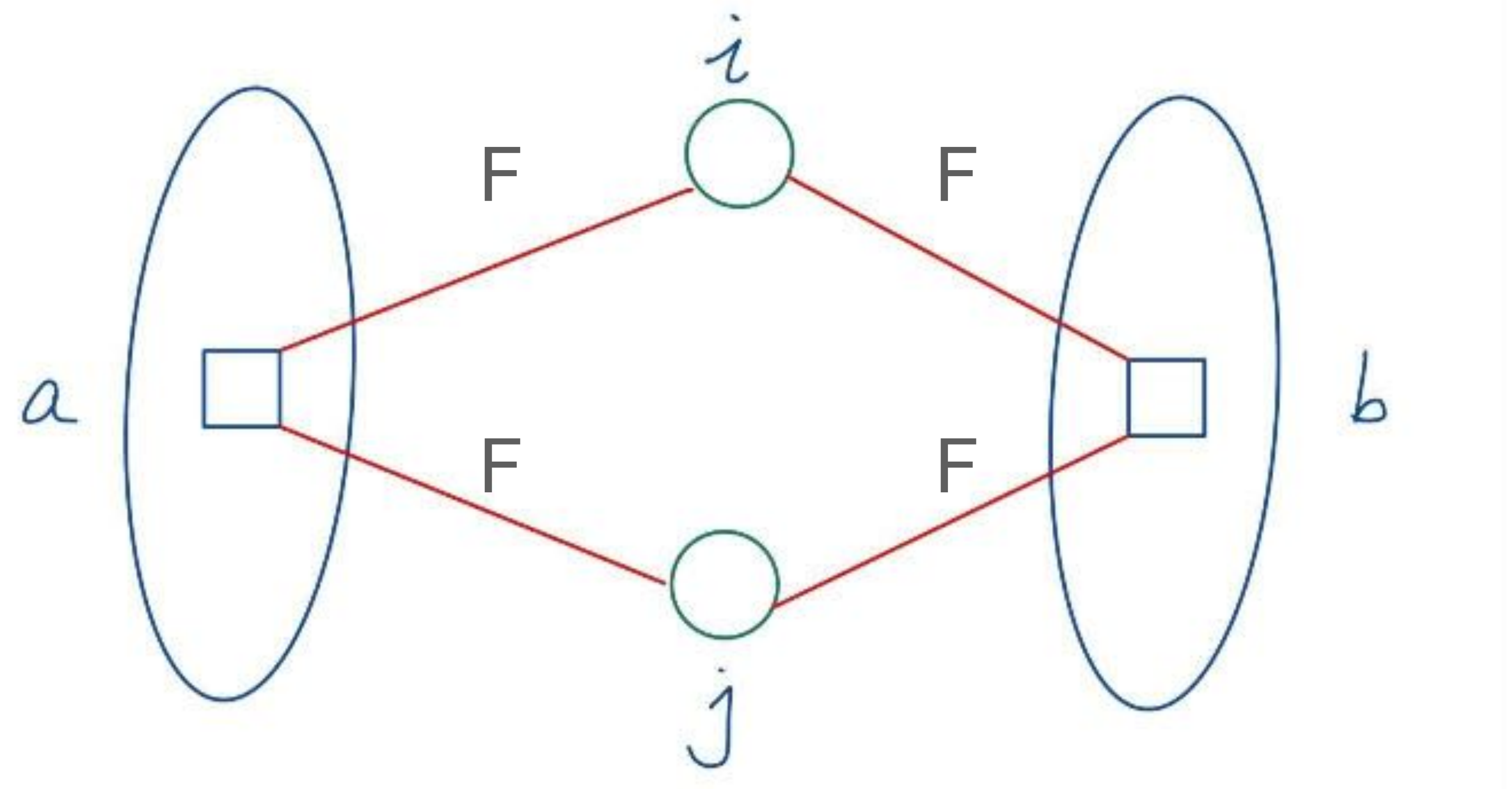

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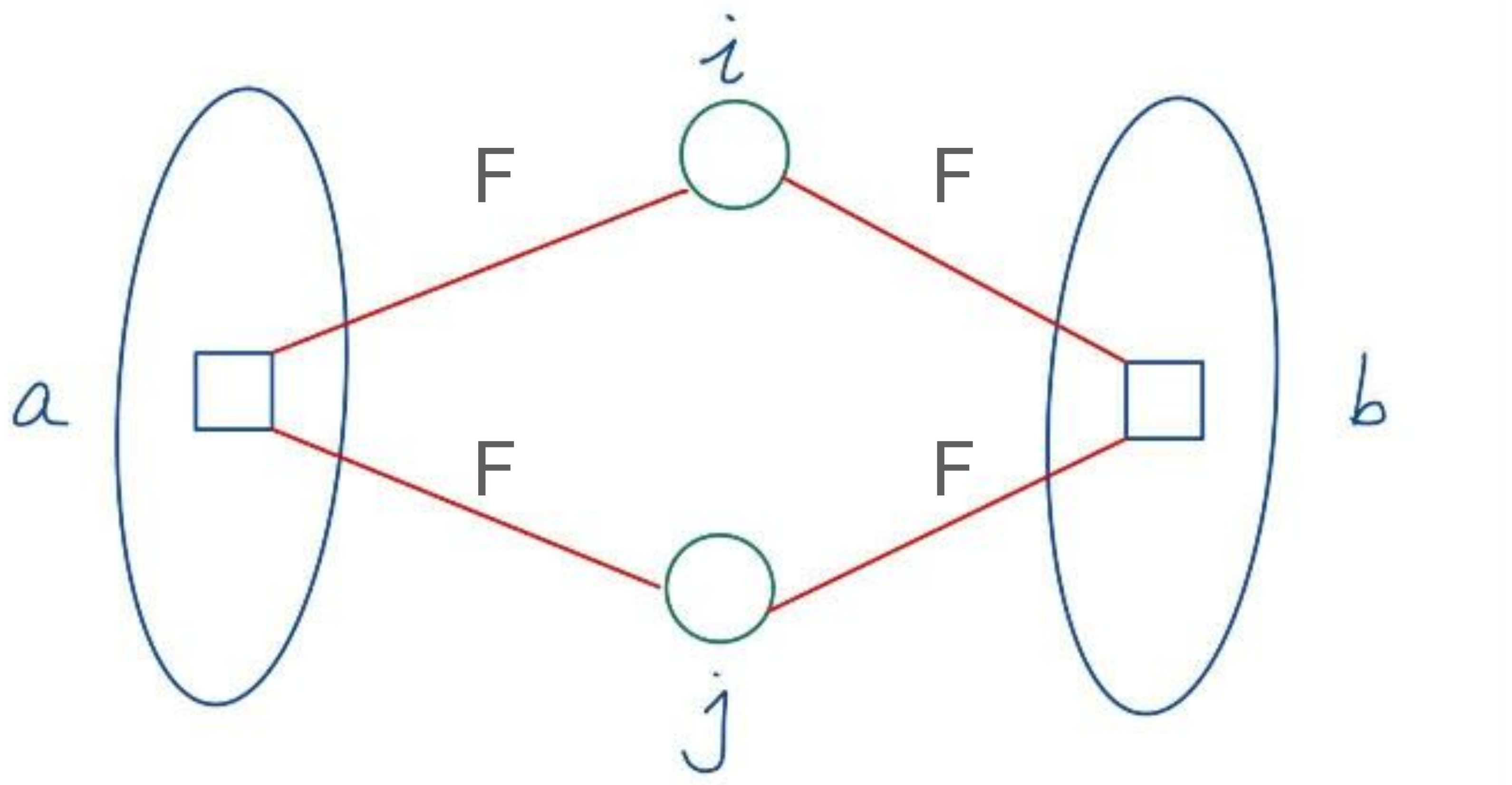

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A Taste of Our Local Analysis

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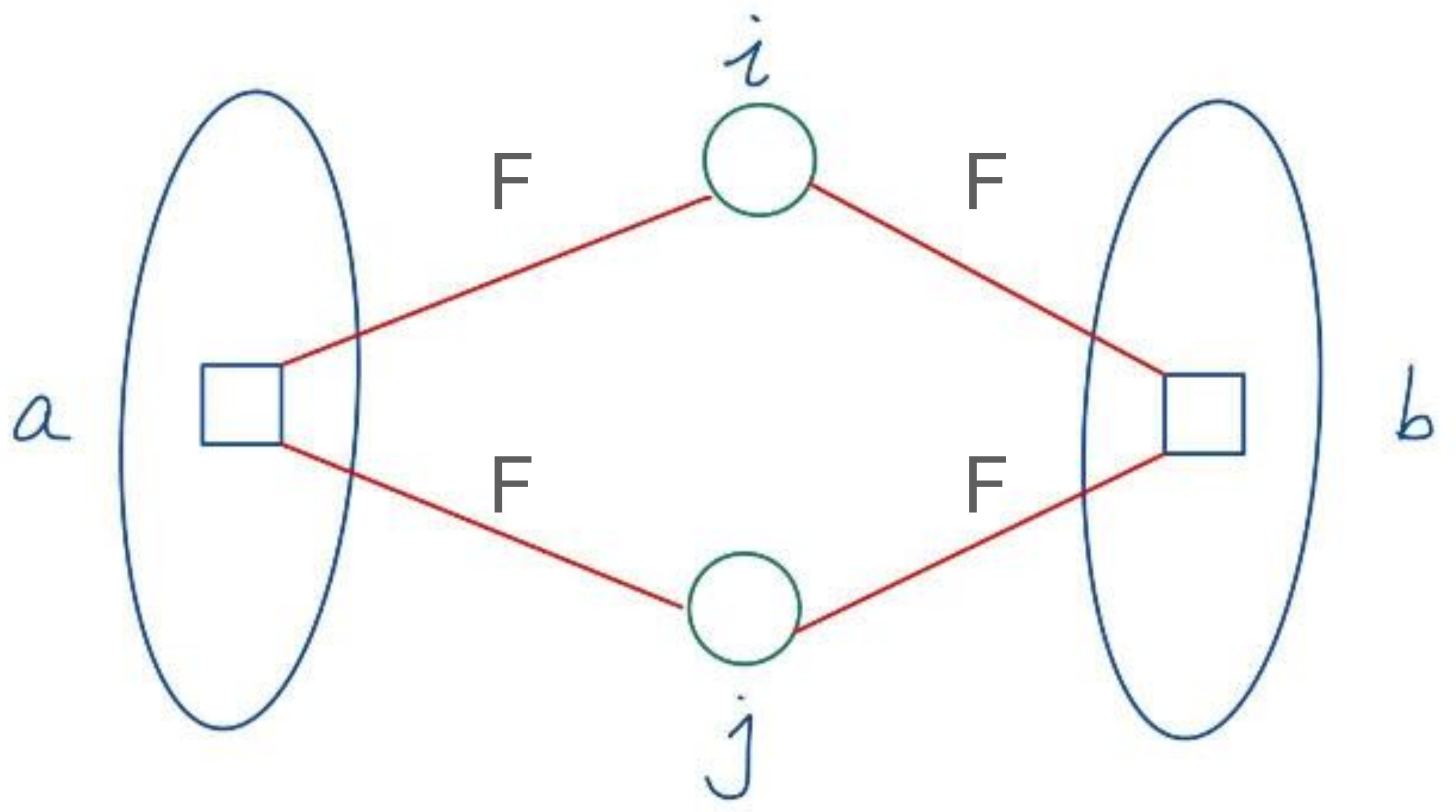

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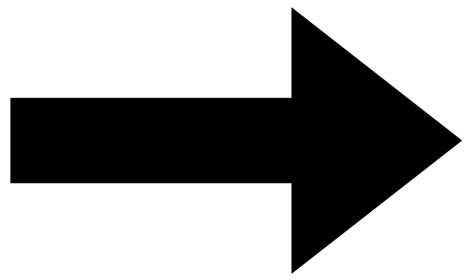

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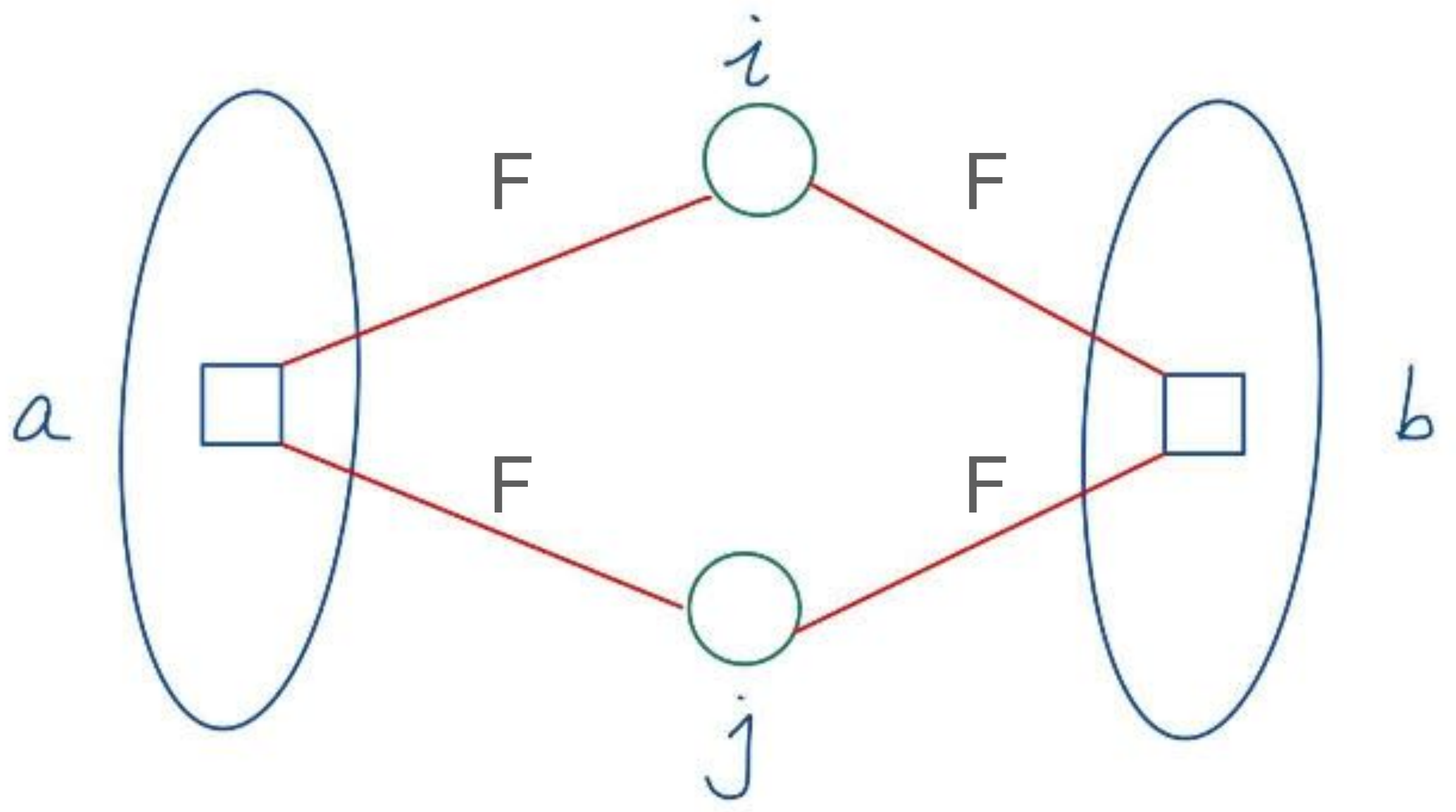

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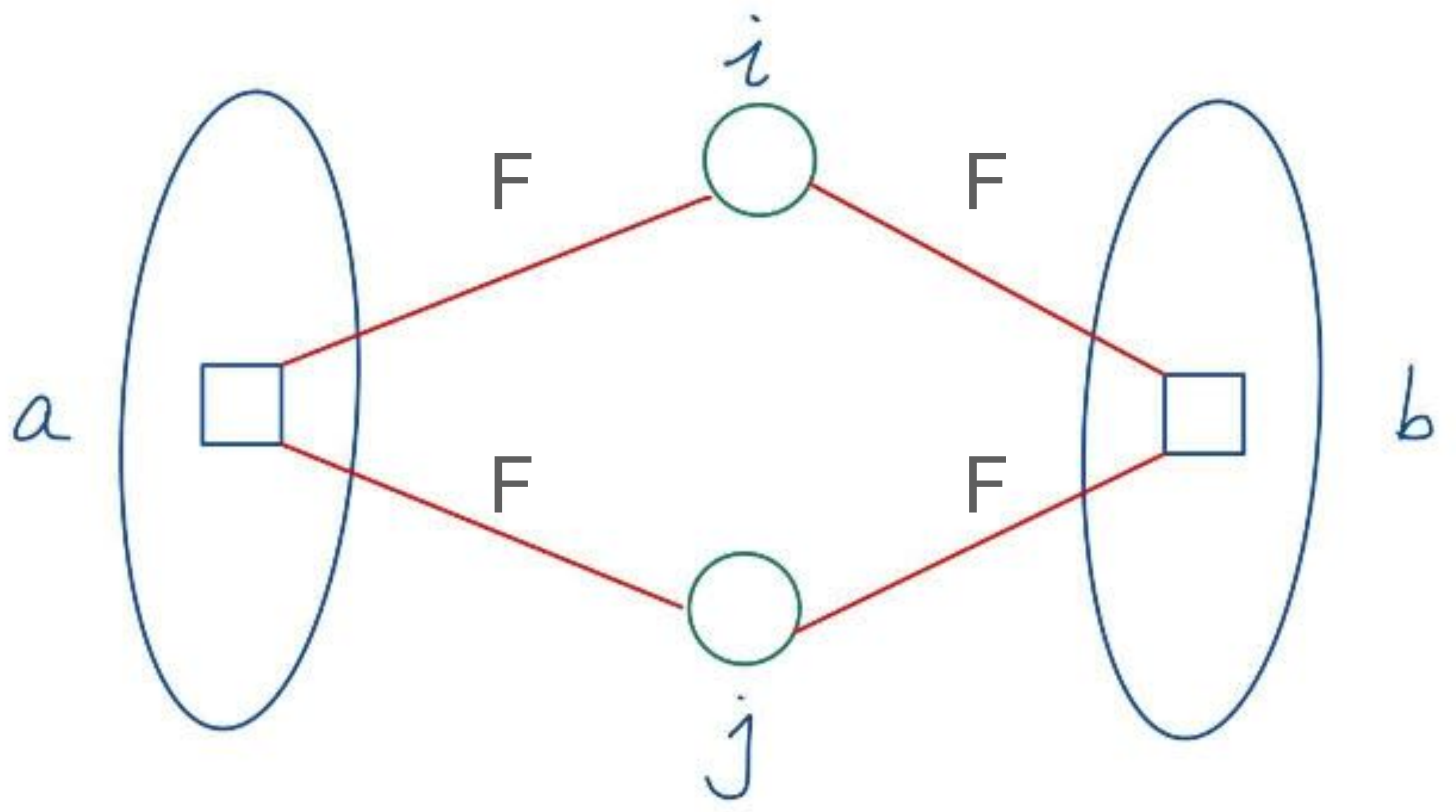

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Dominant term!

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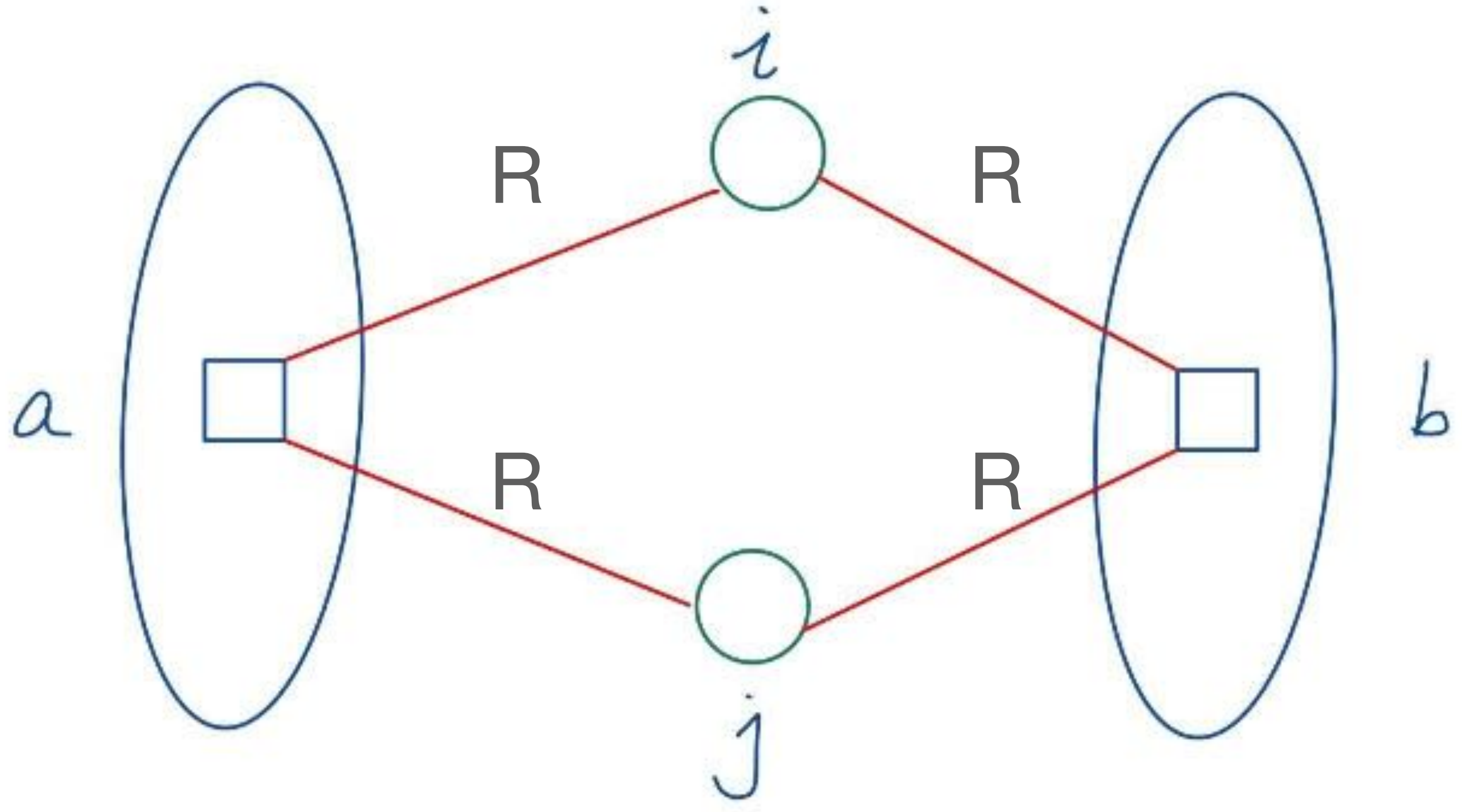

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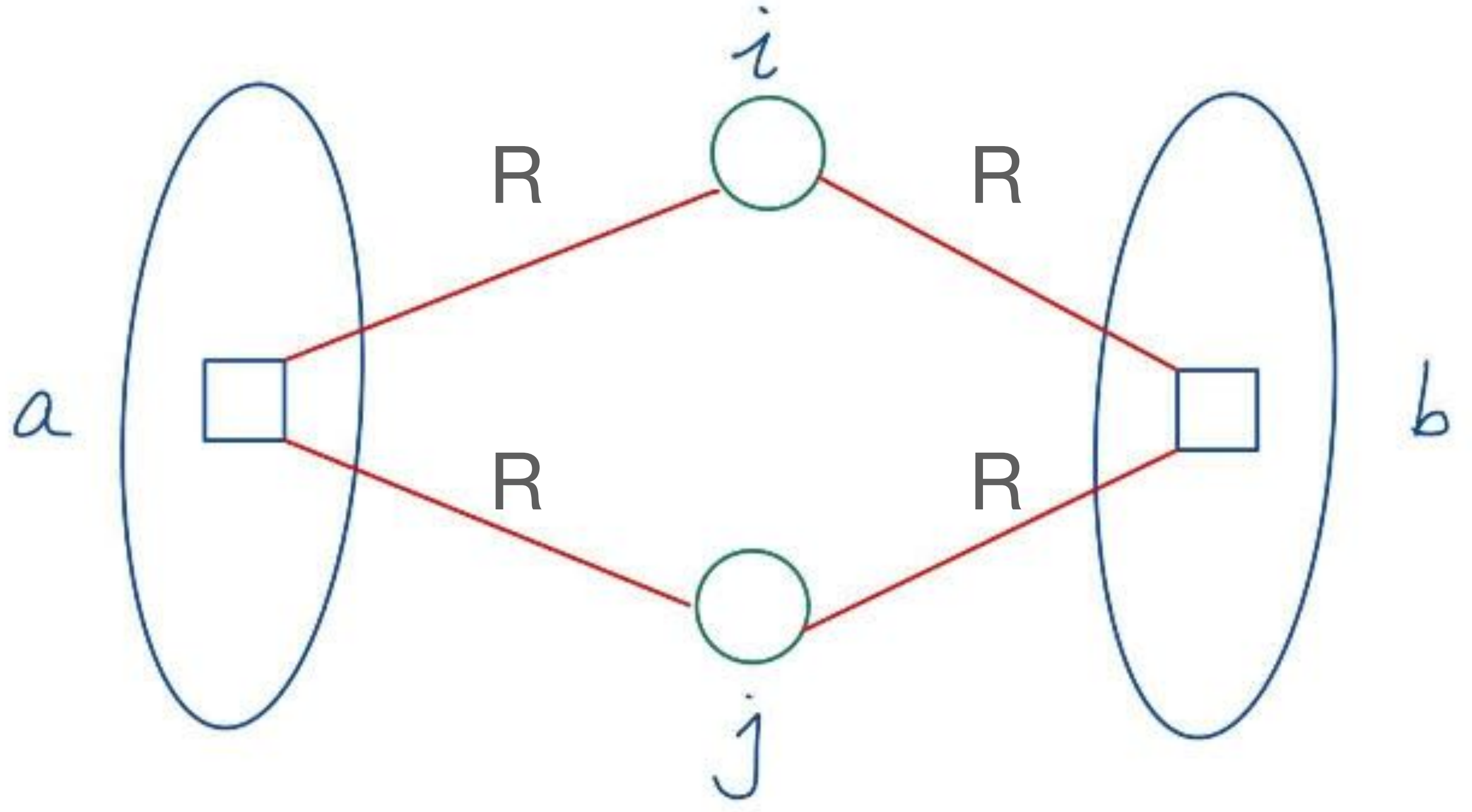

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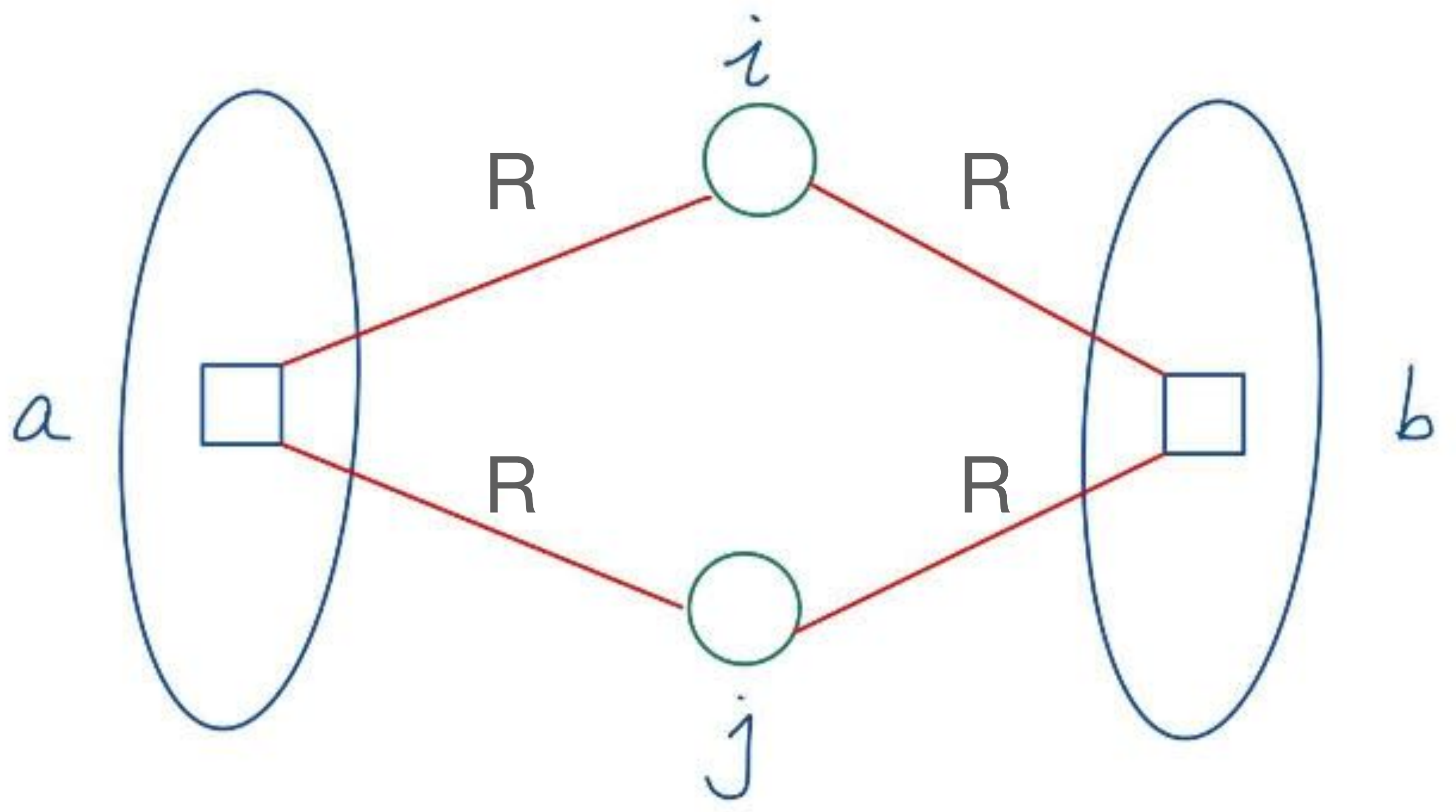

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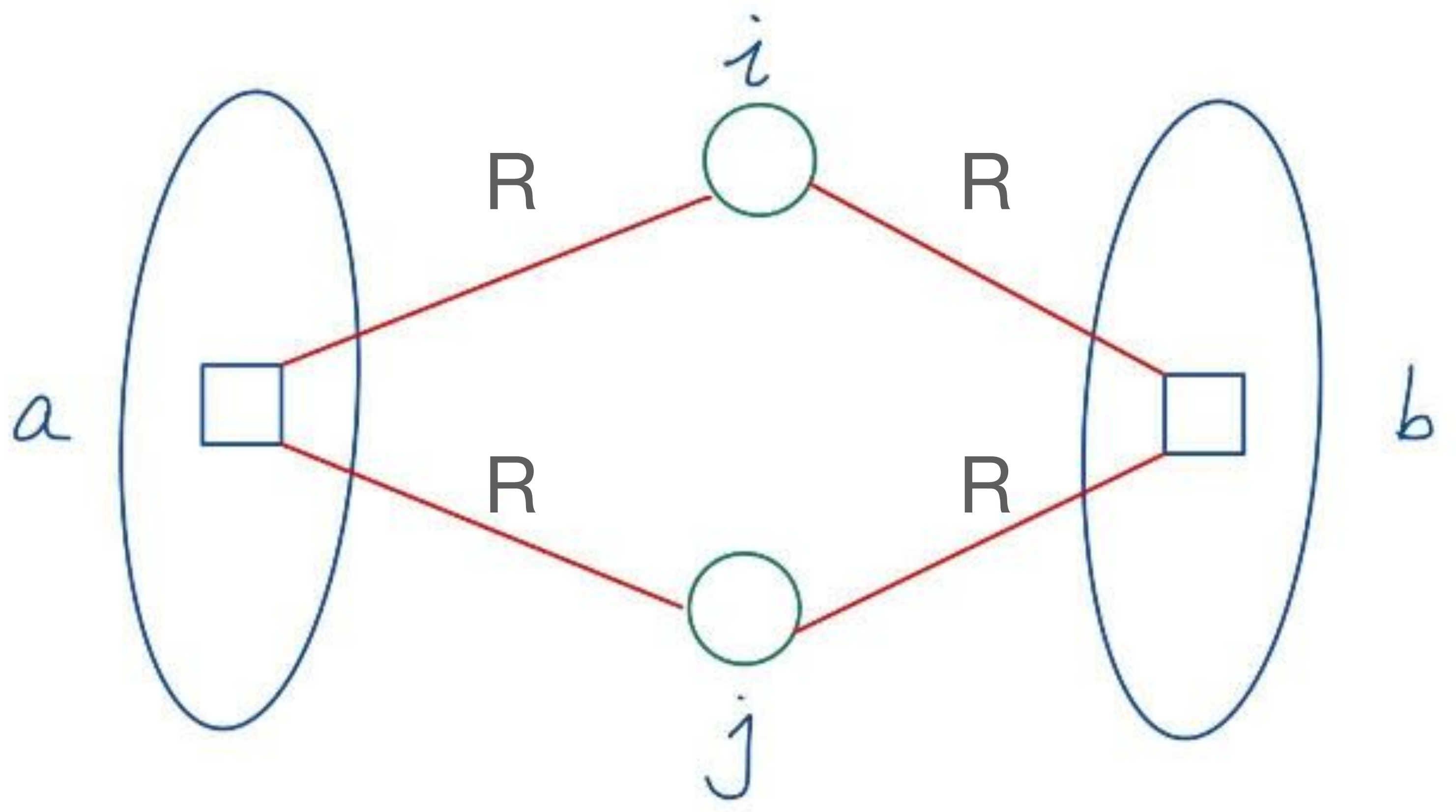

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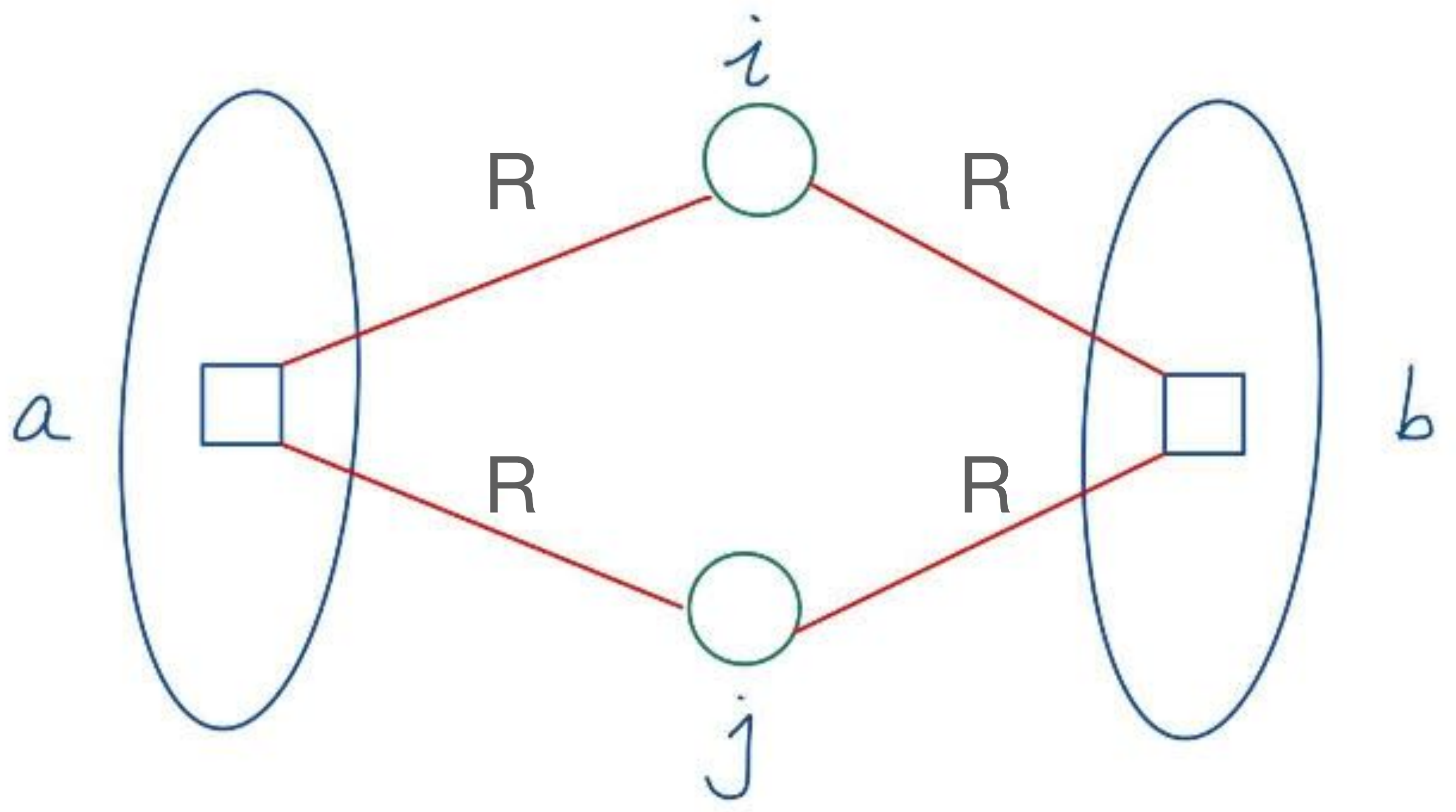

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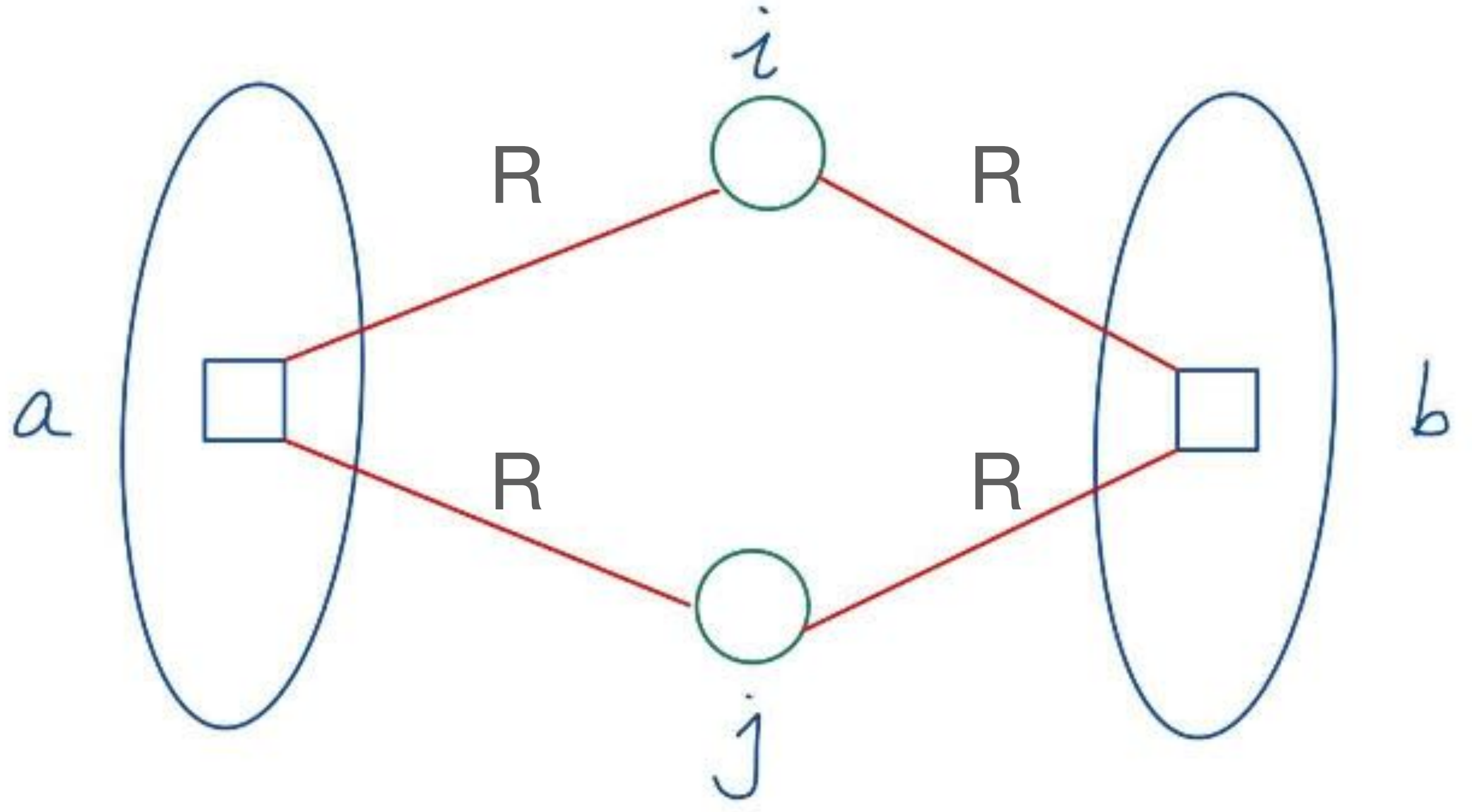

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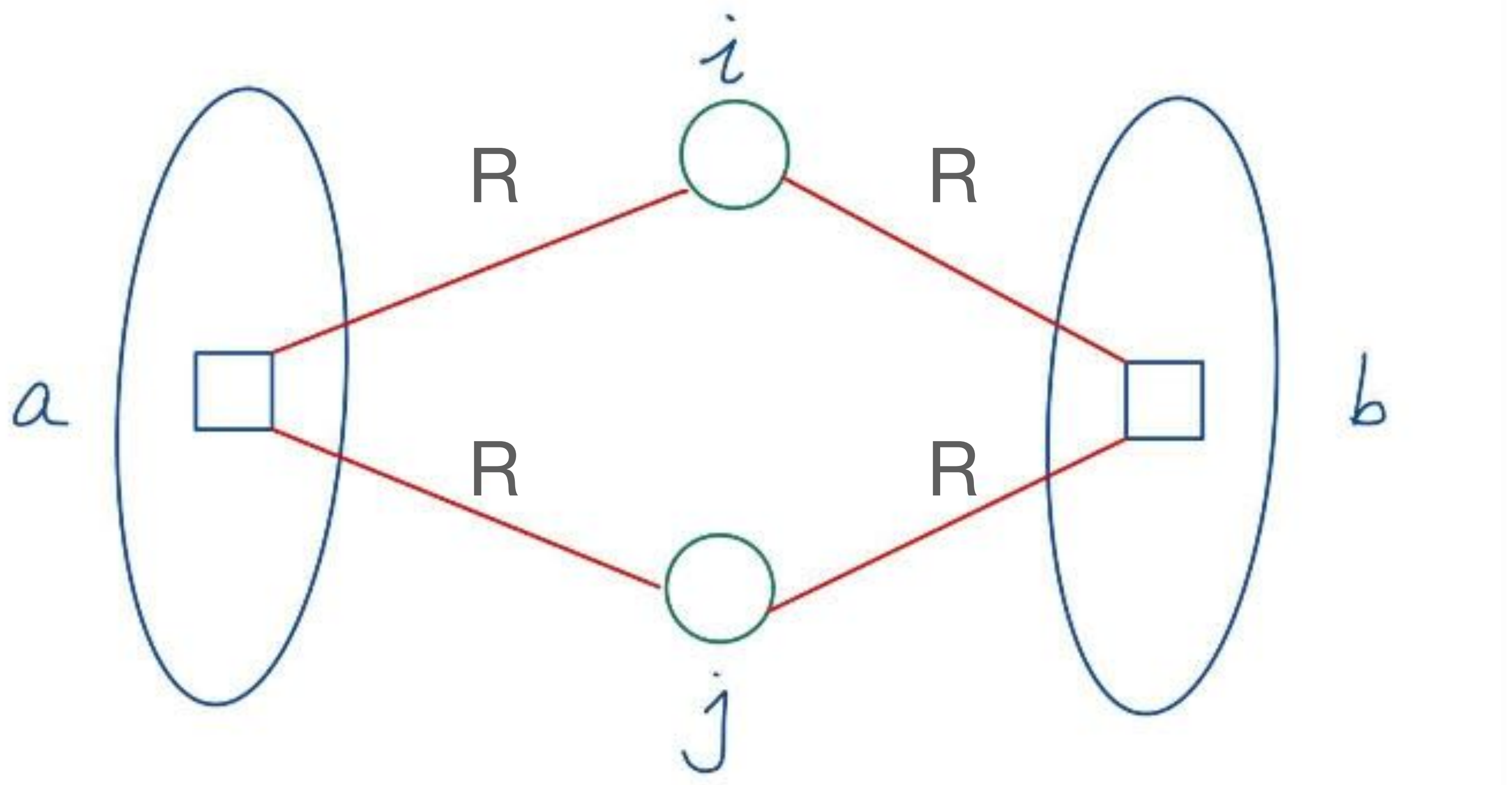

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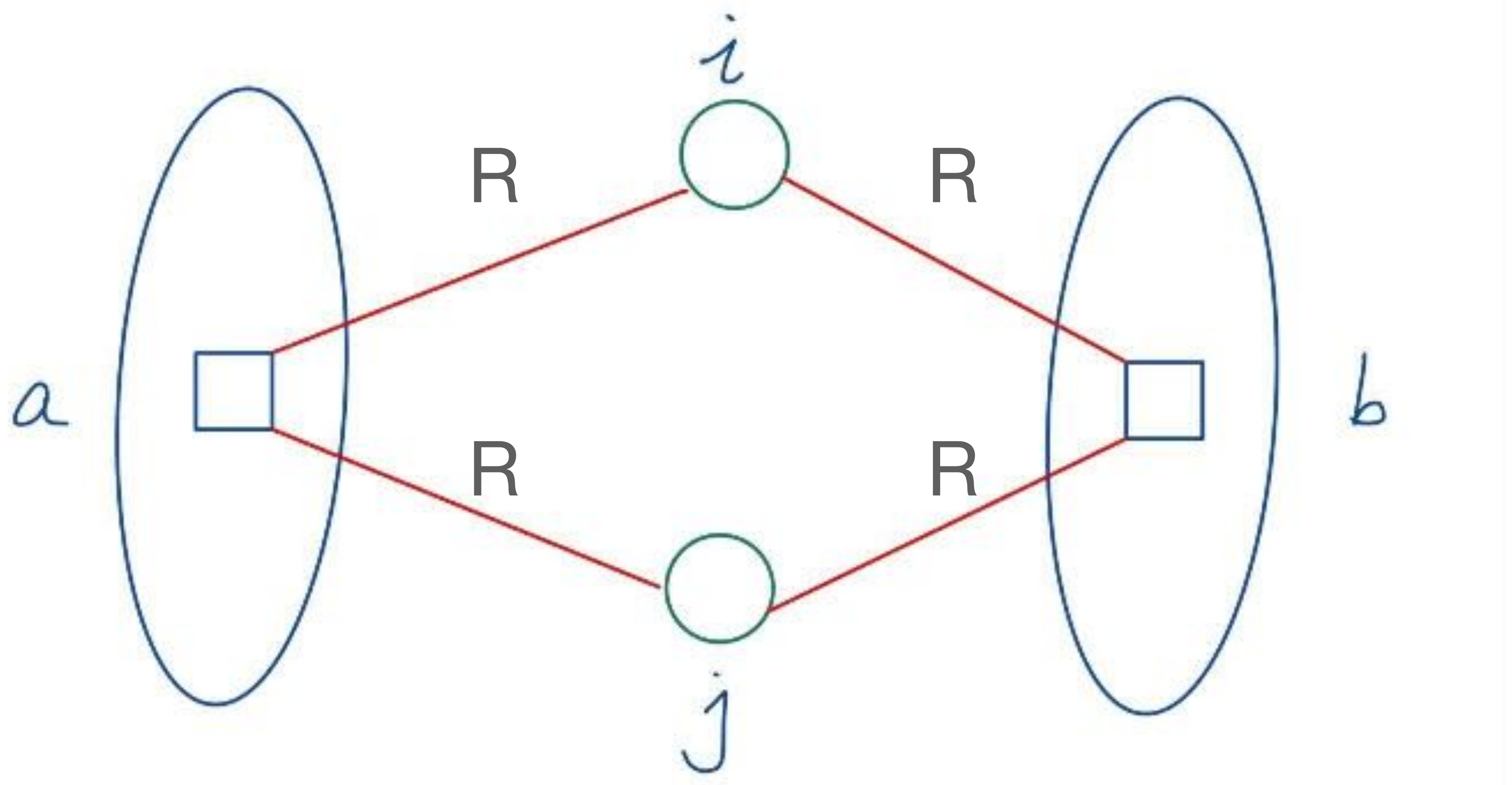

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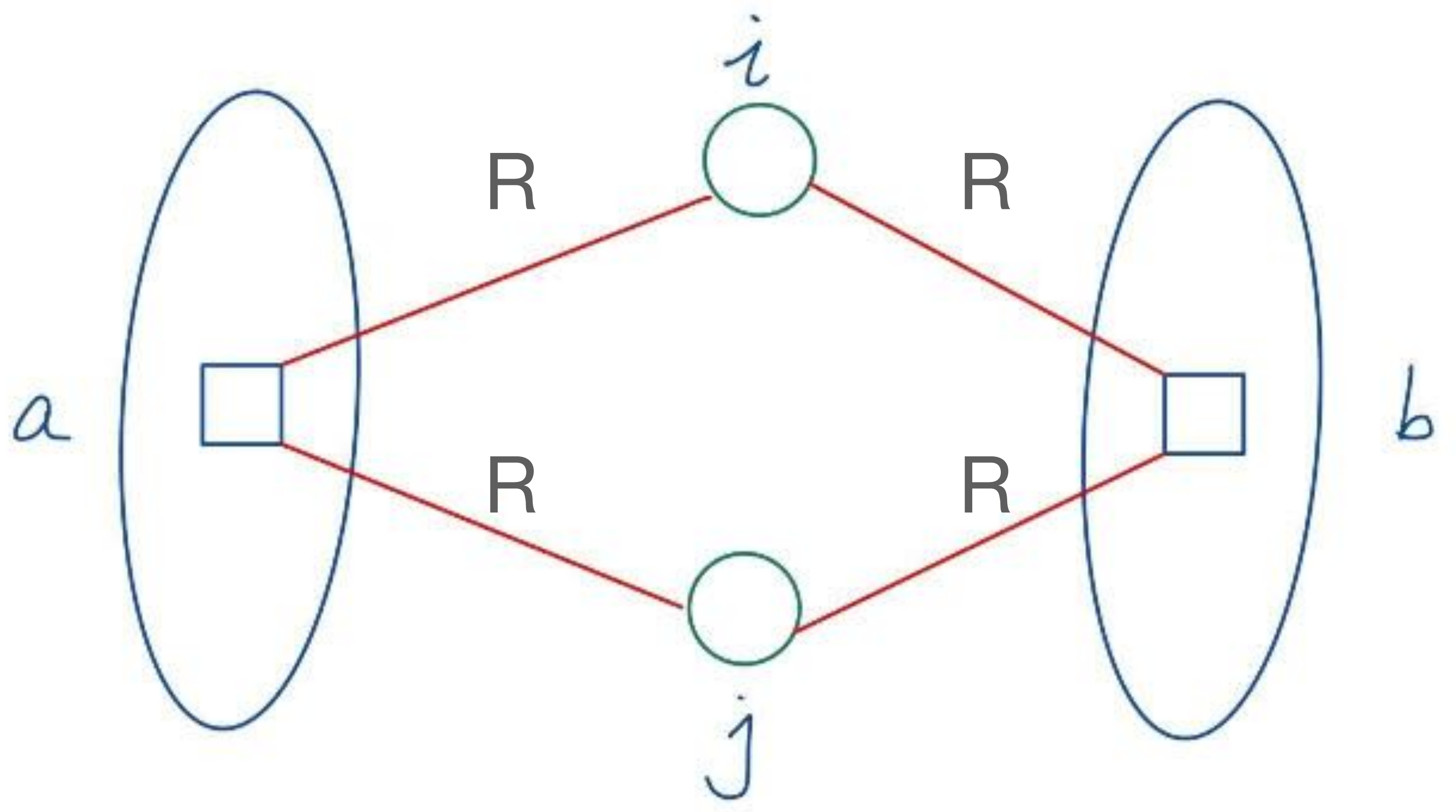

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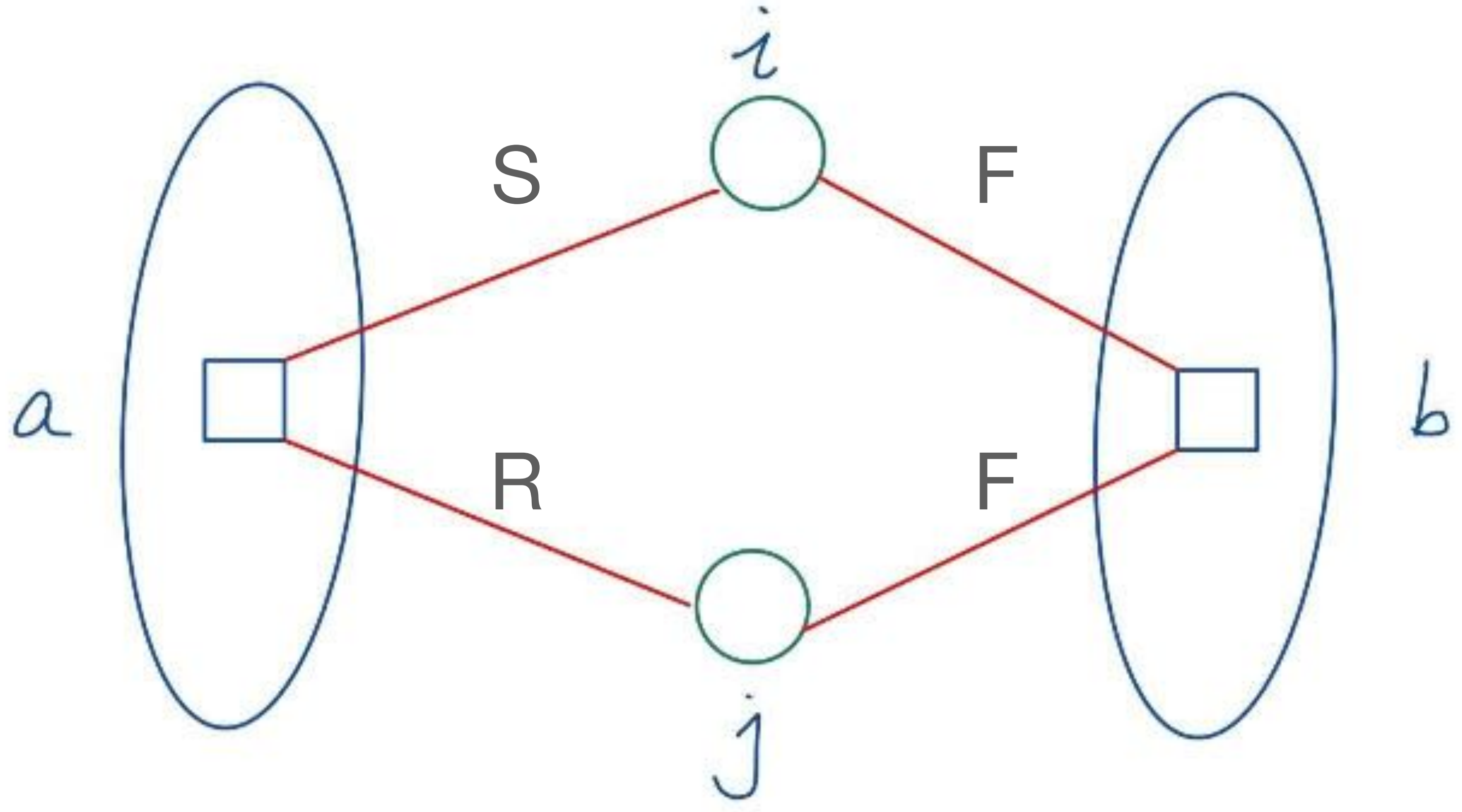

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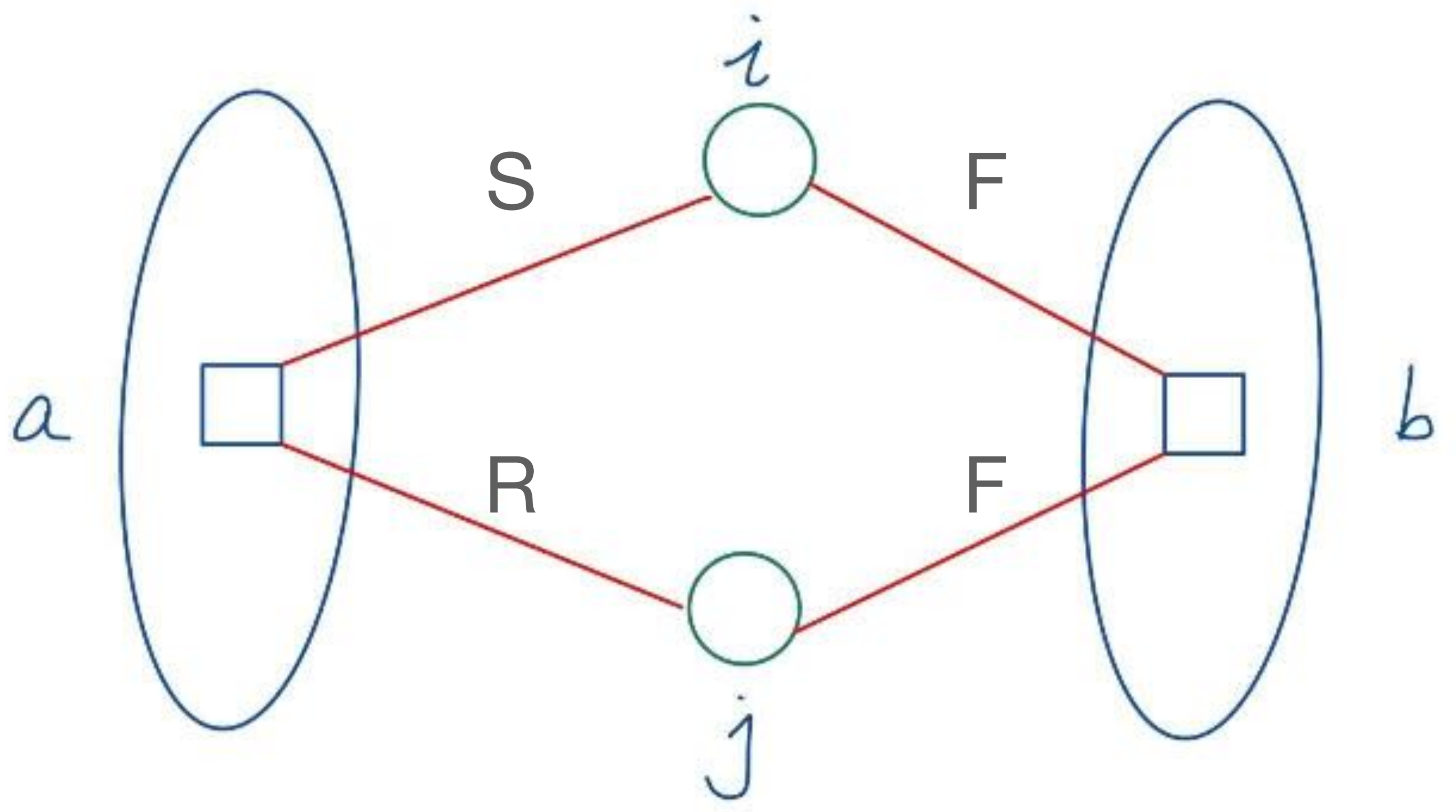

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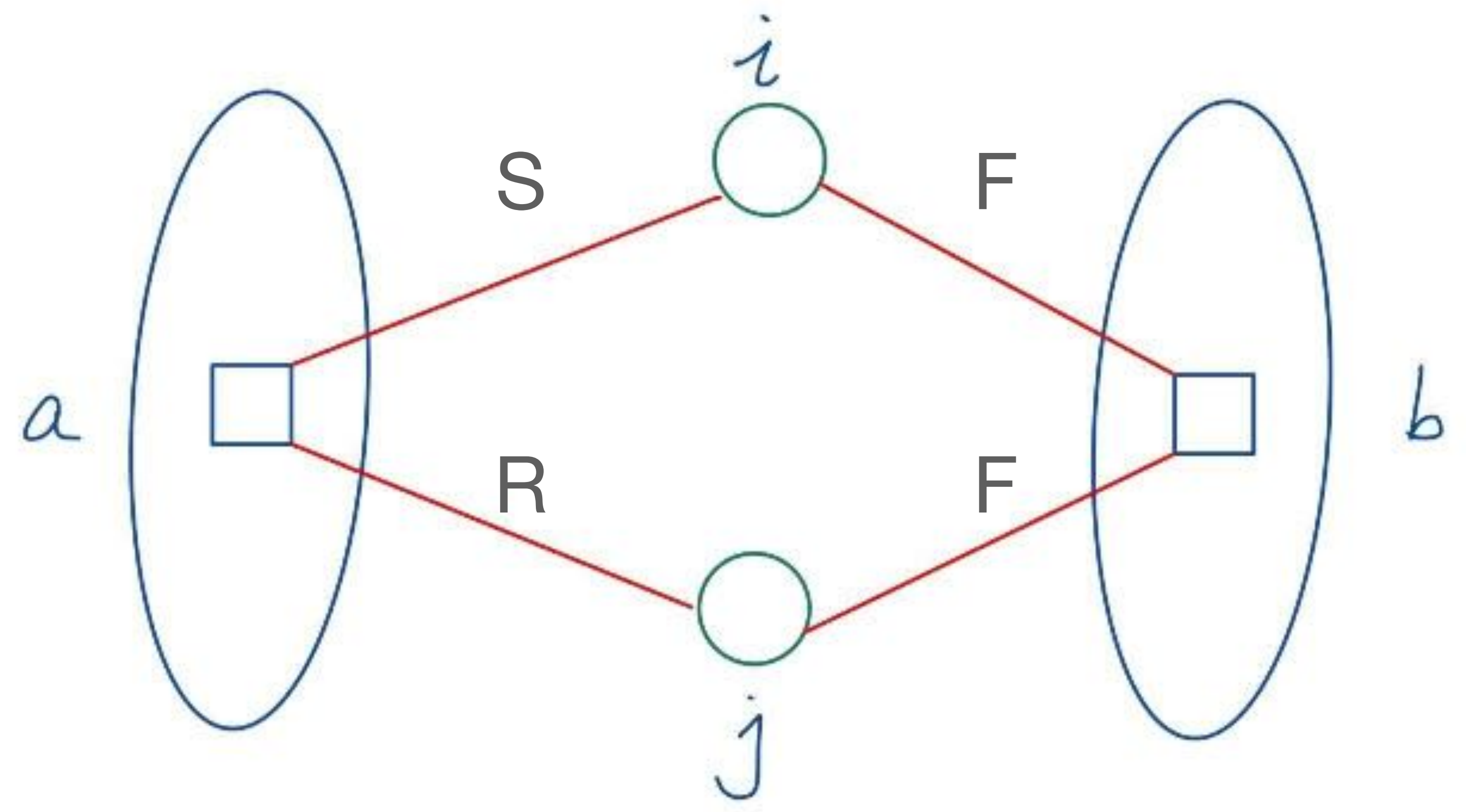

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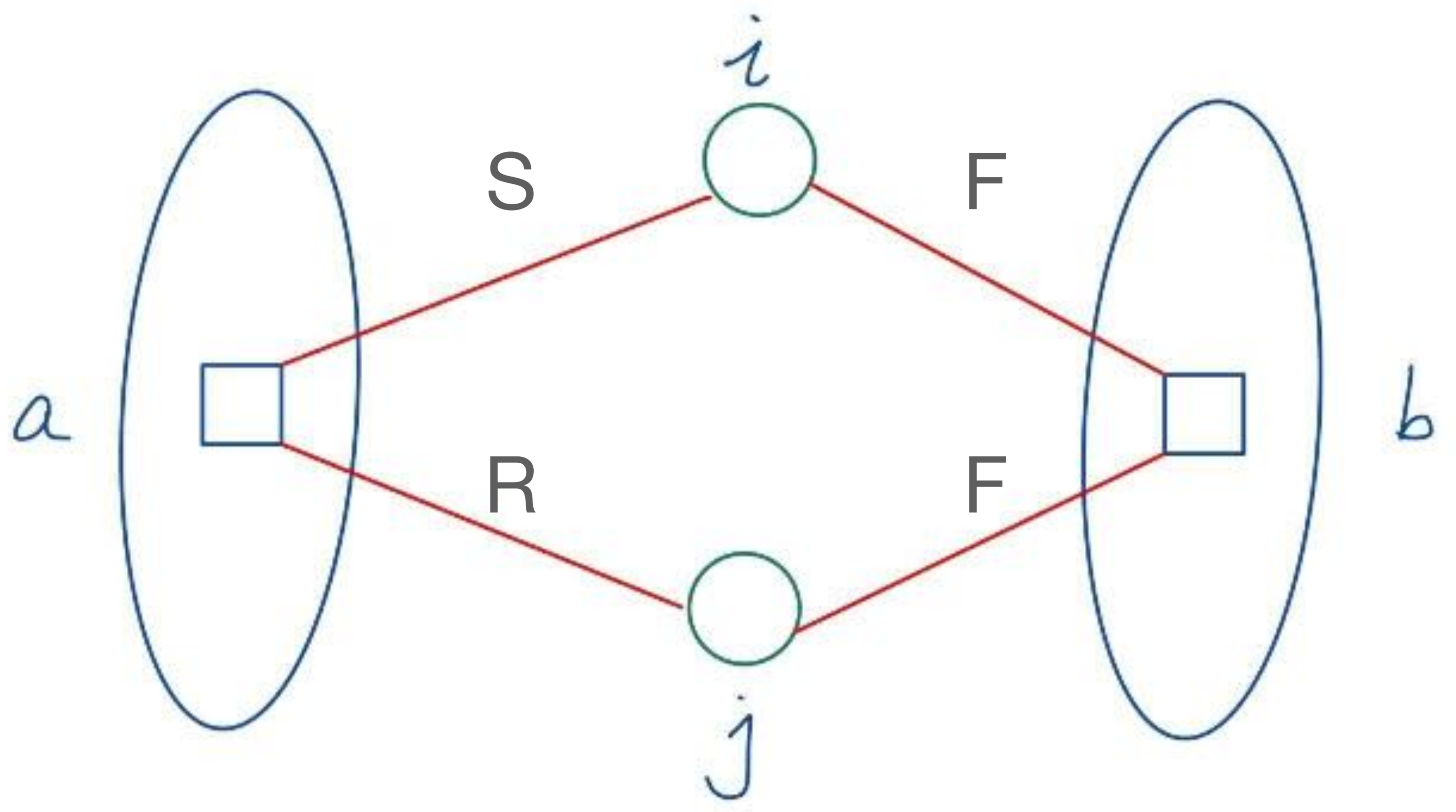

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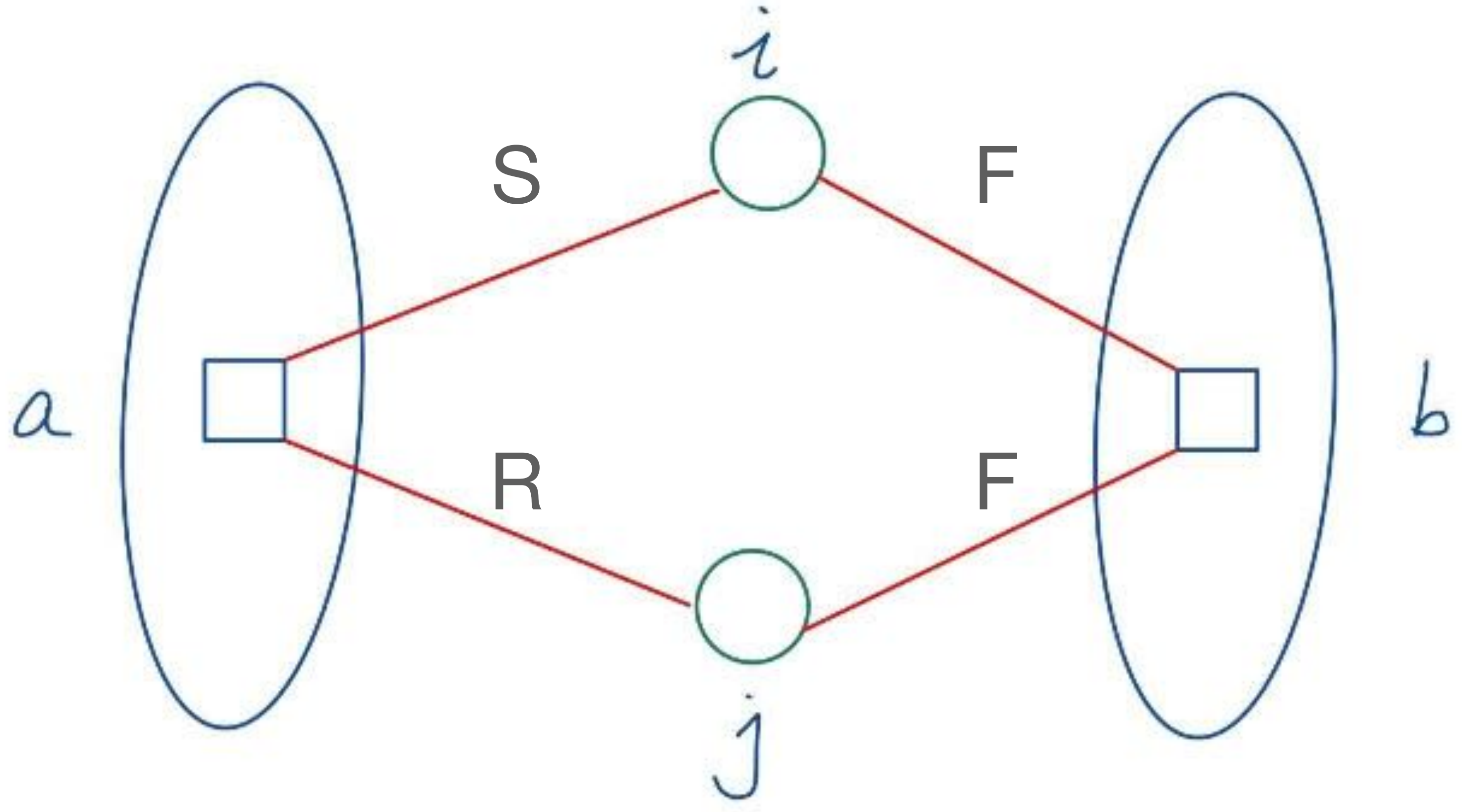

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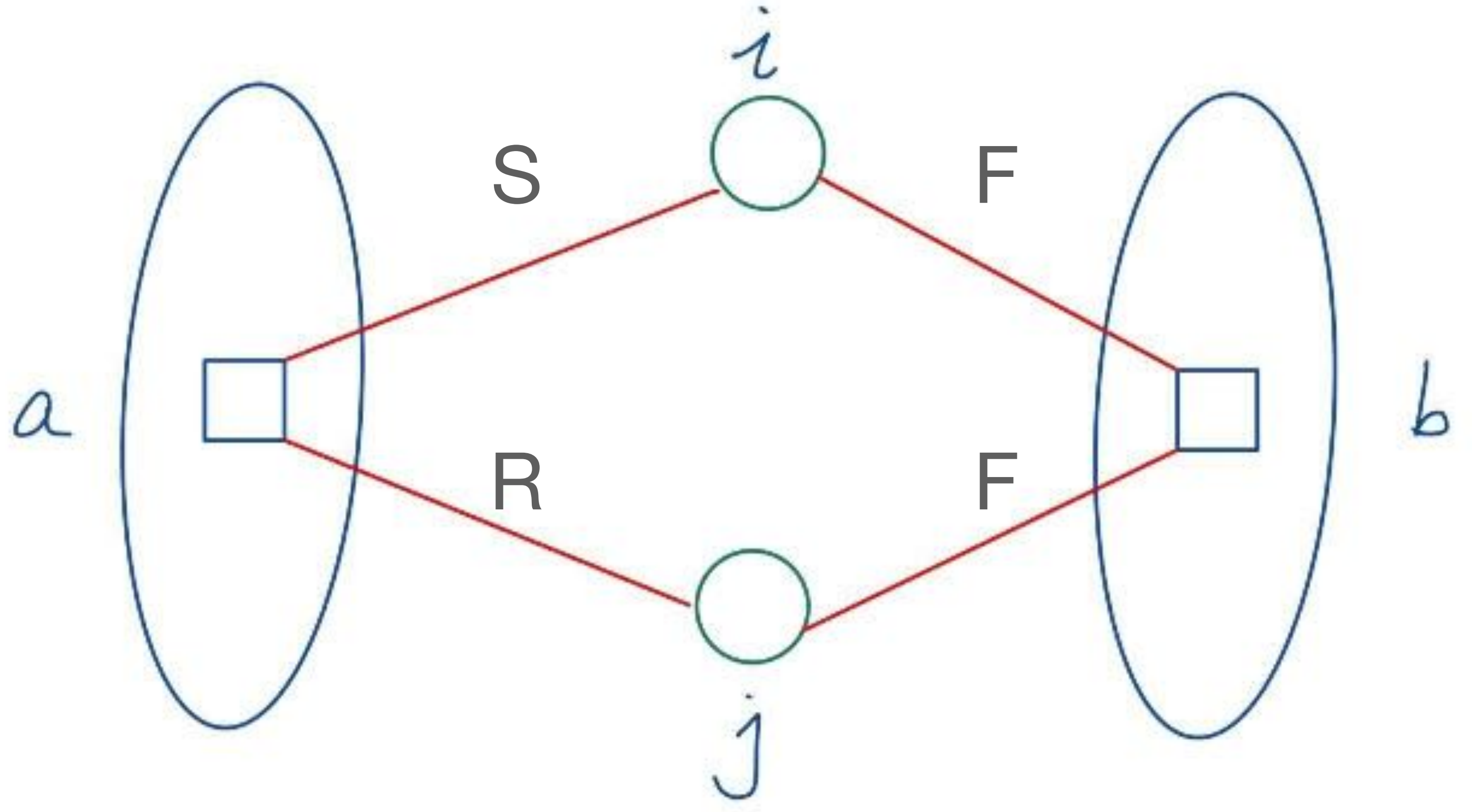

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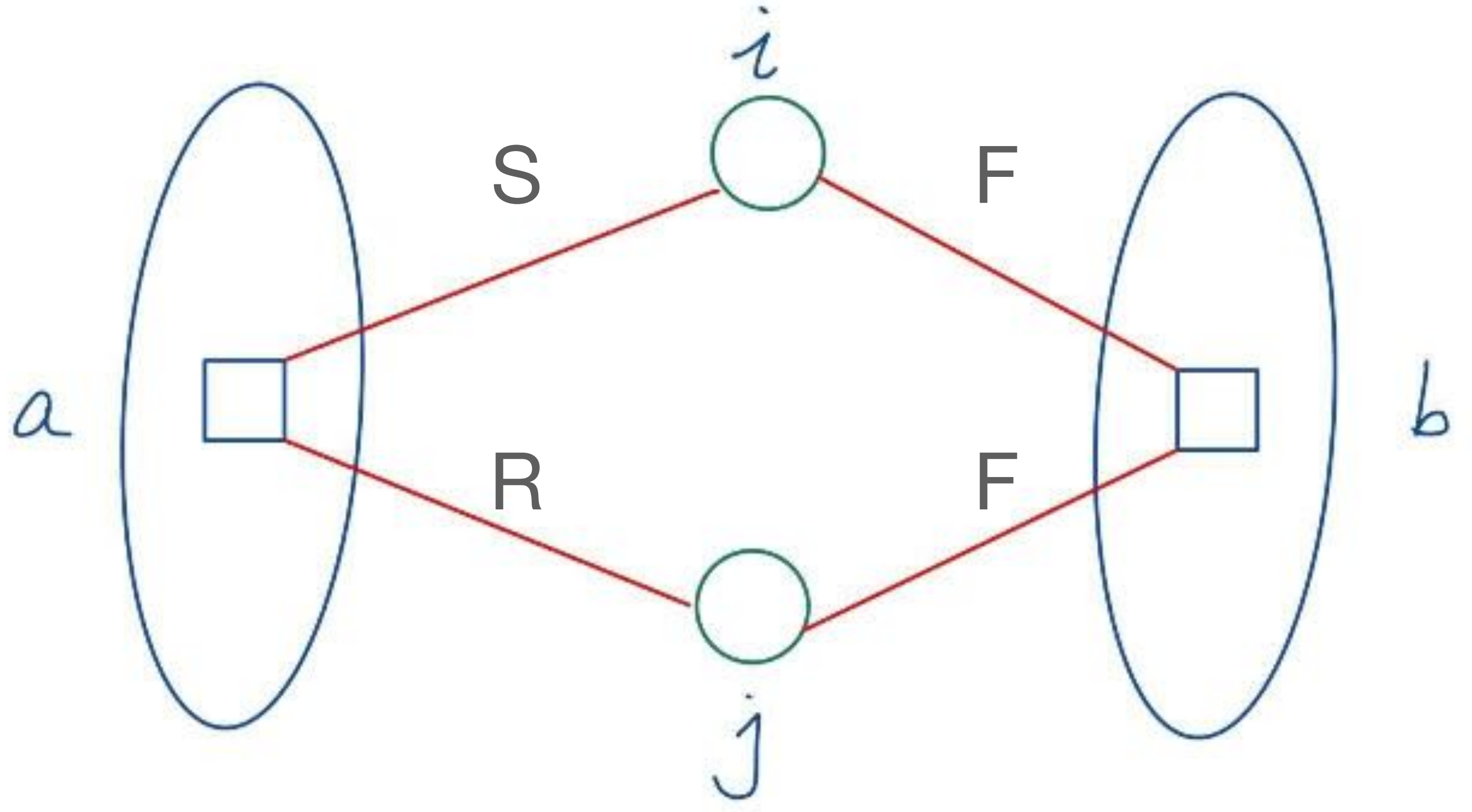

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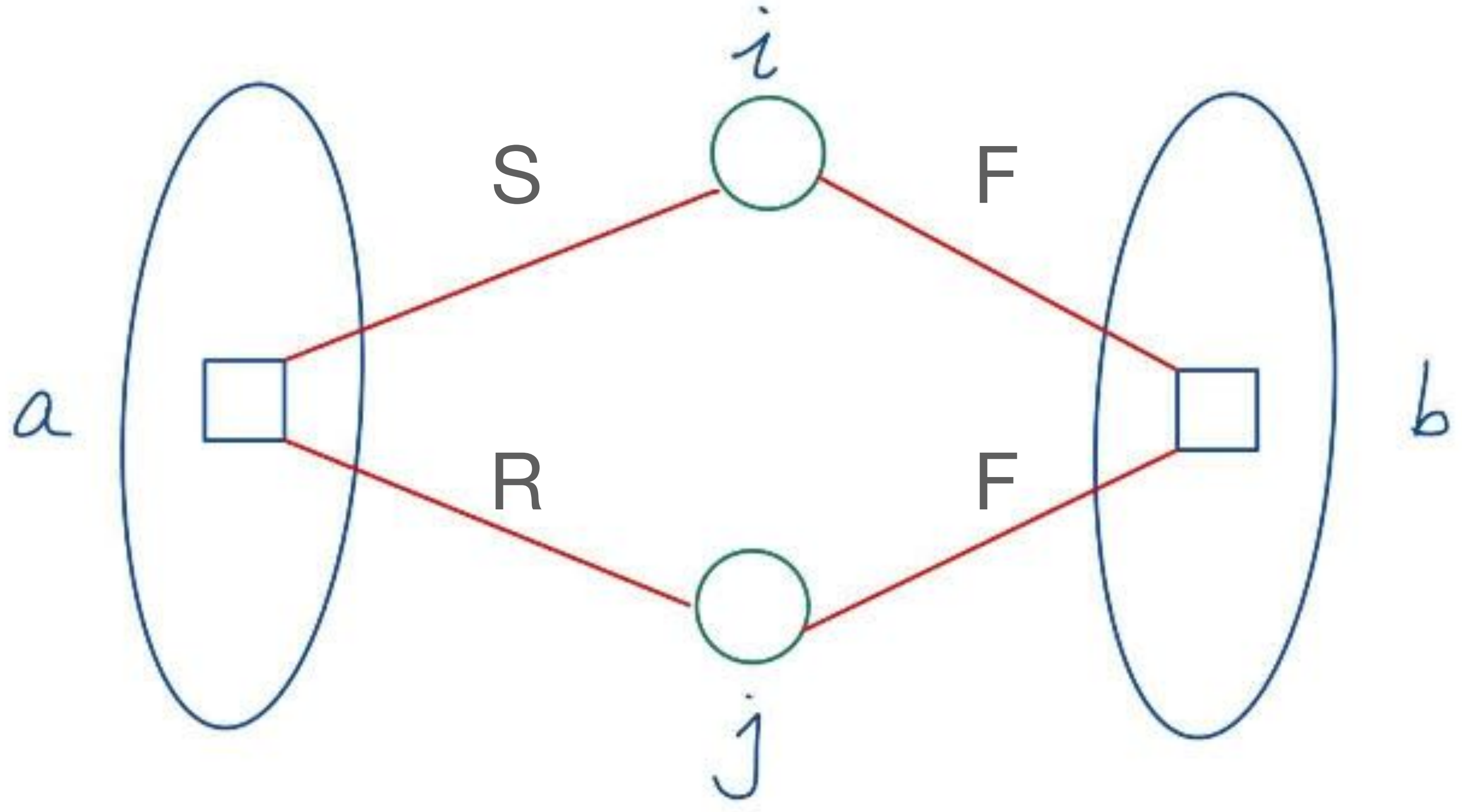

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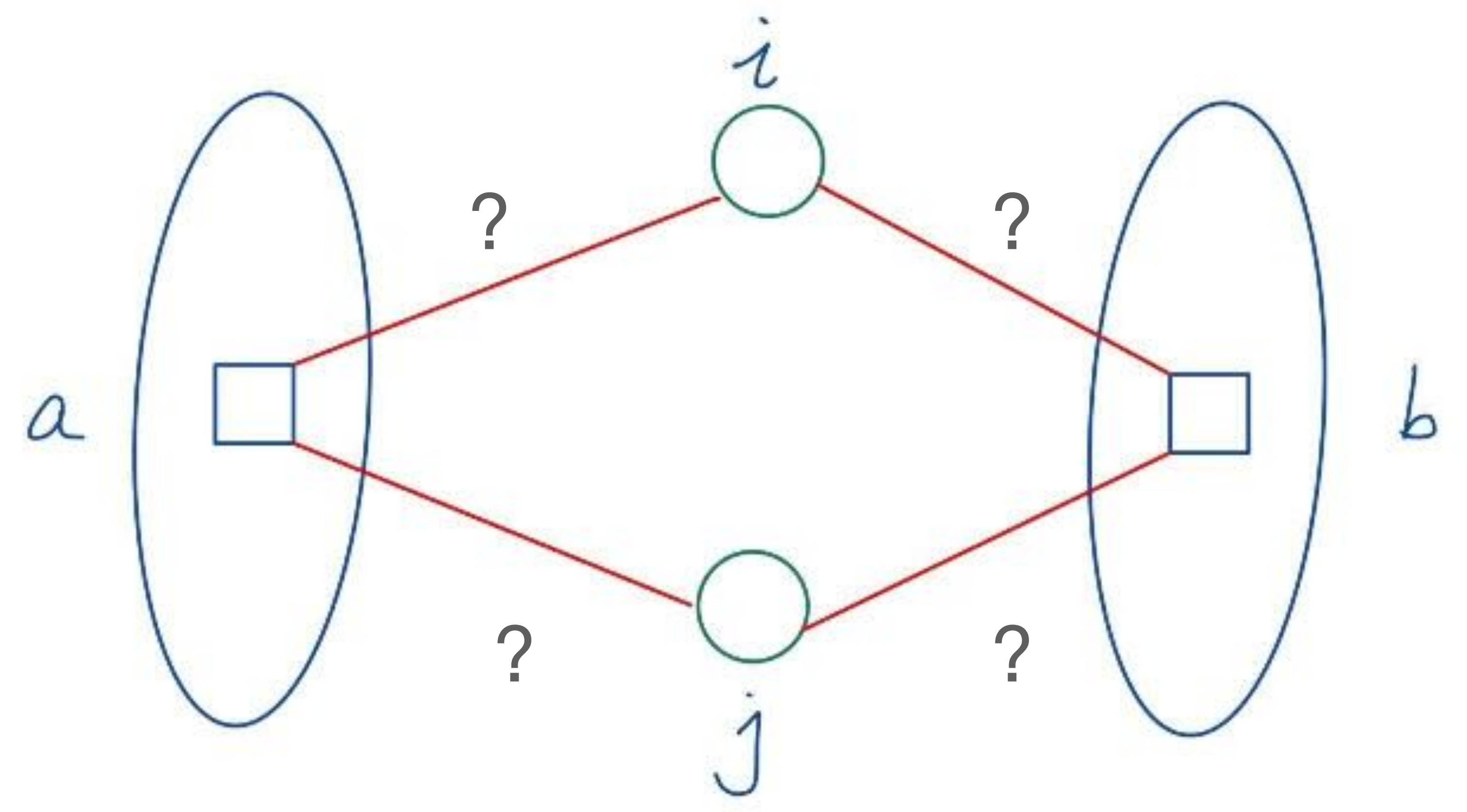

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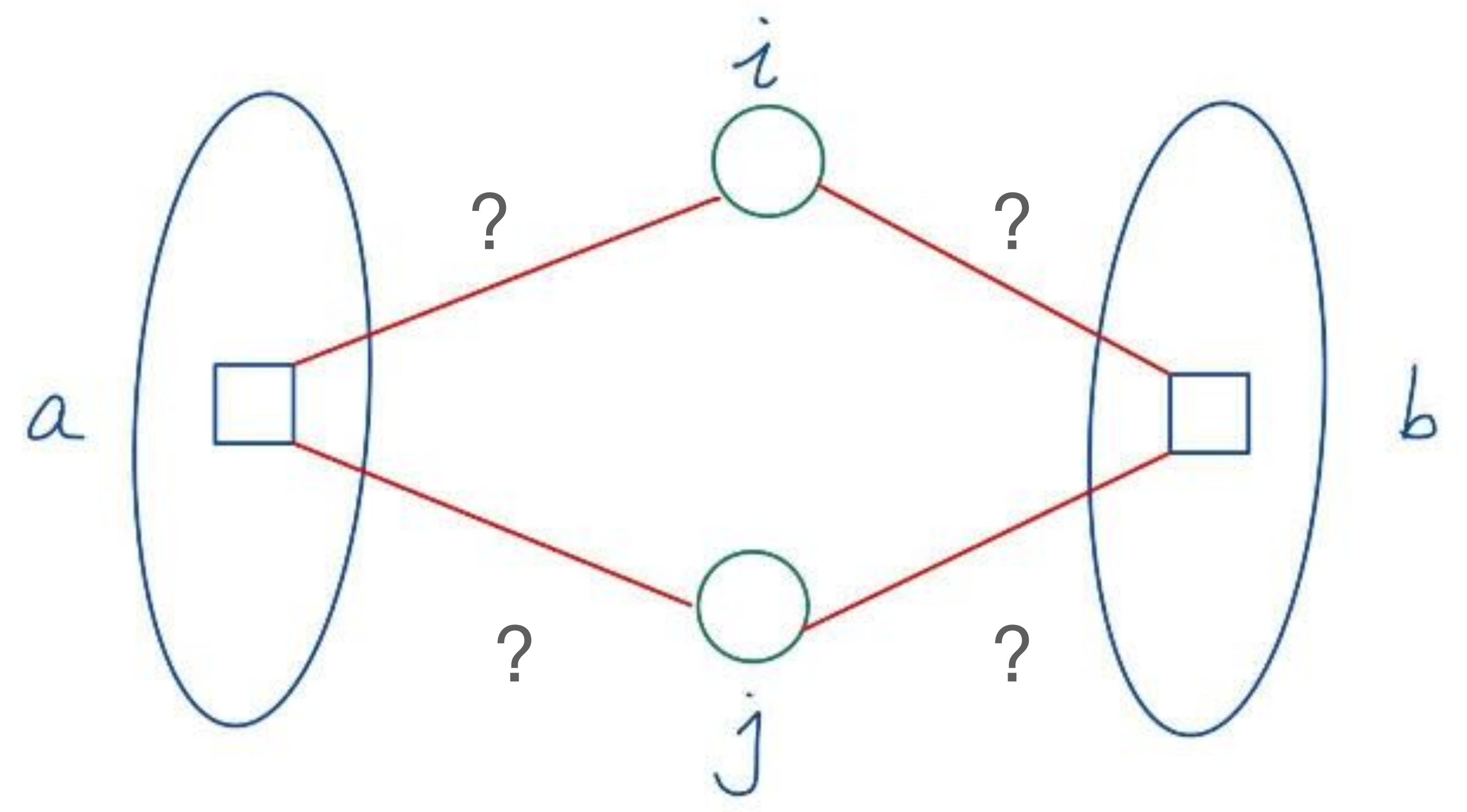

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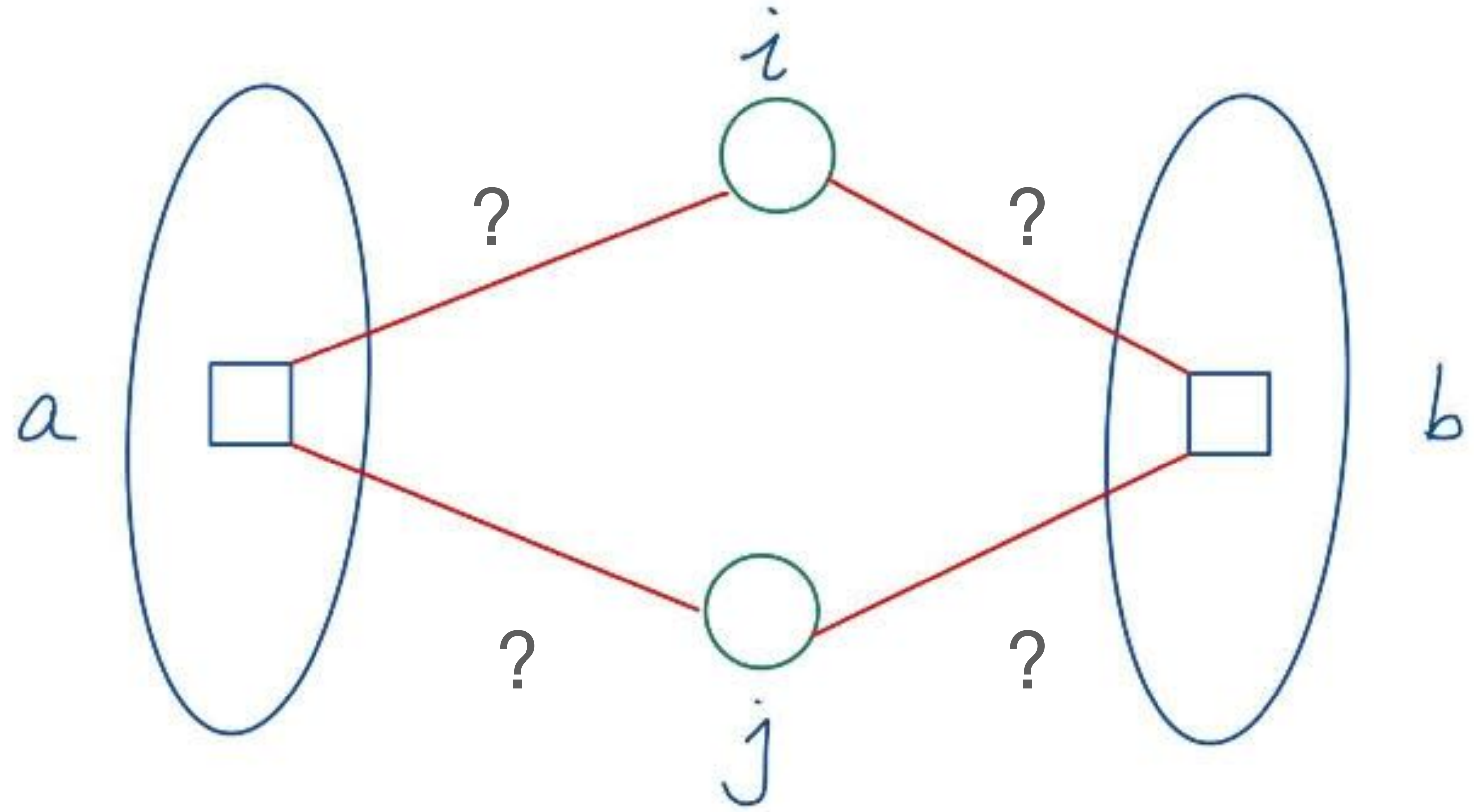

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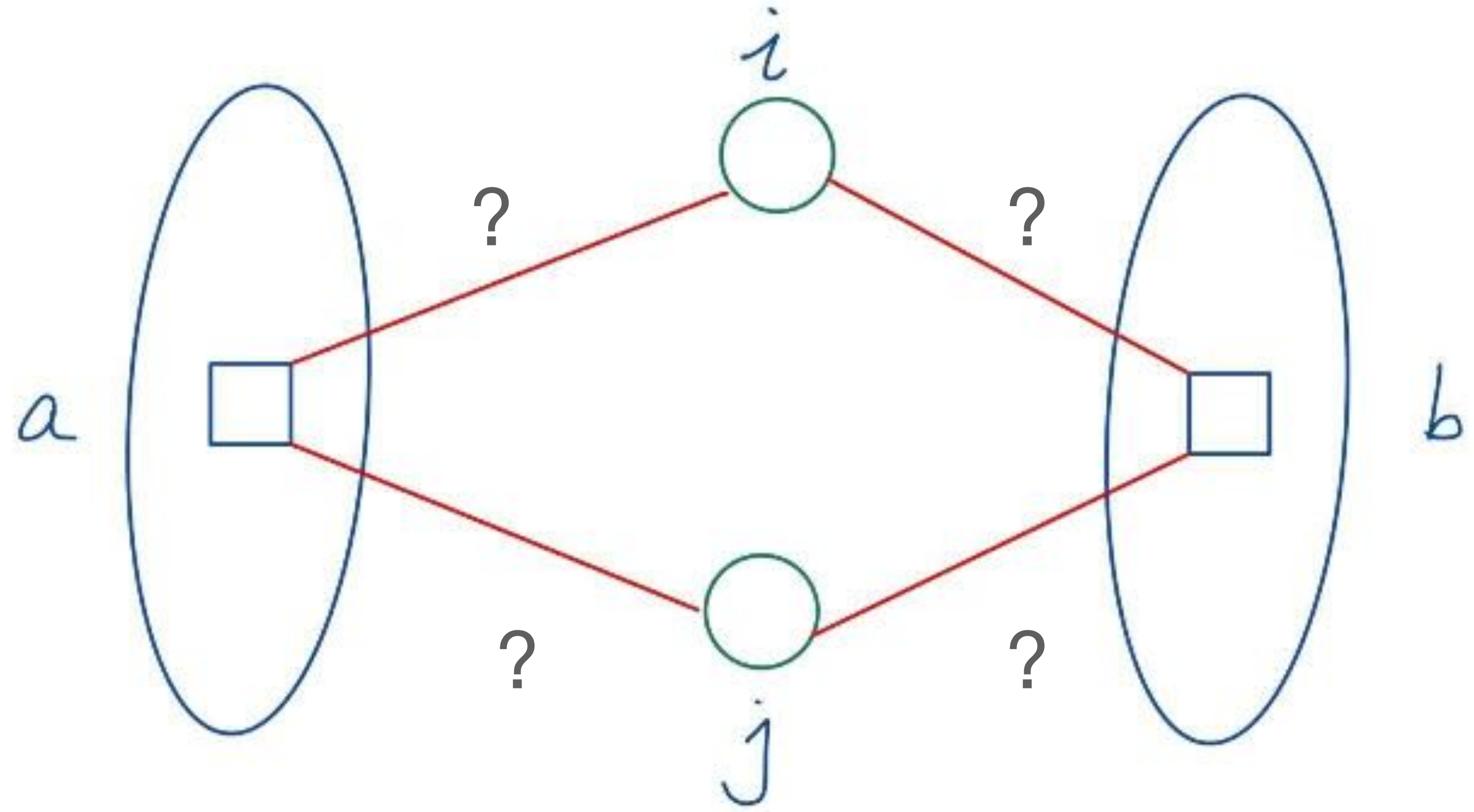

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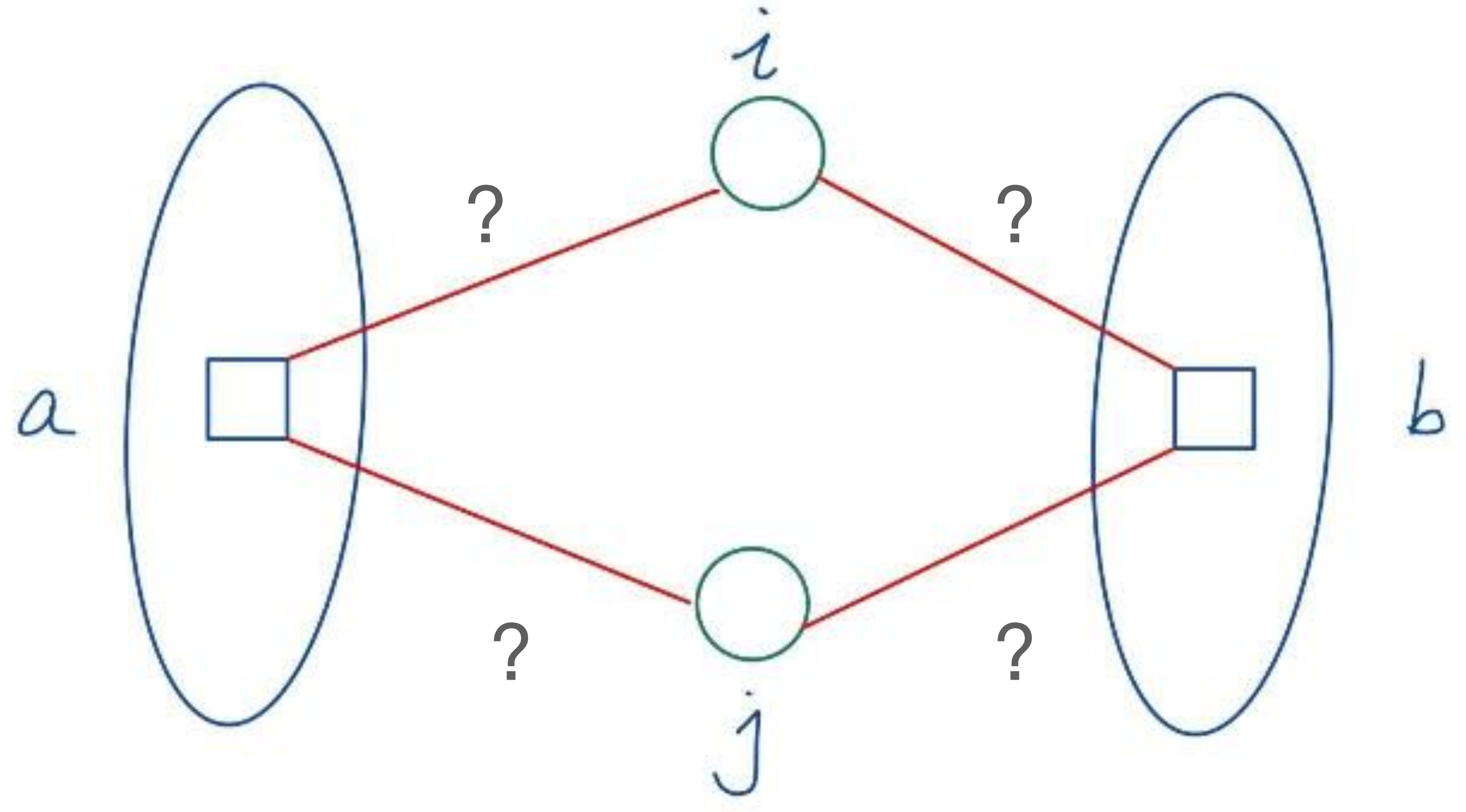

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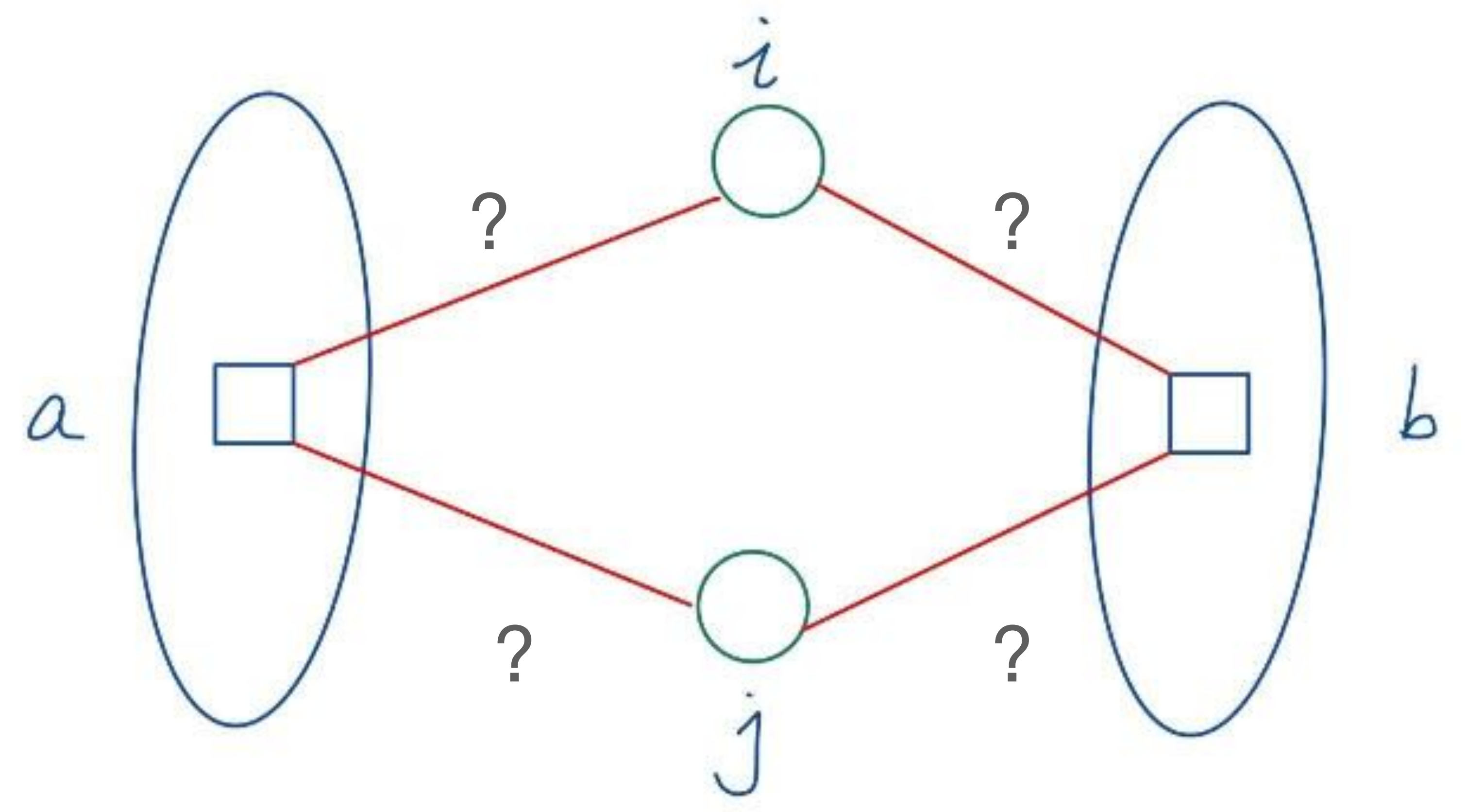

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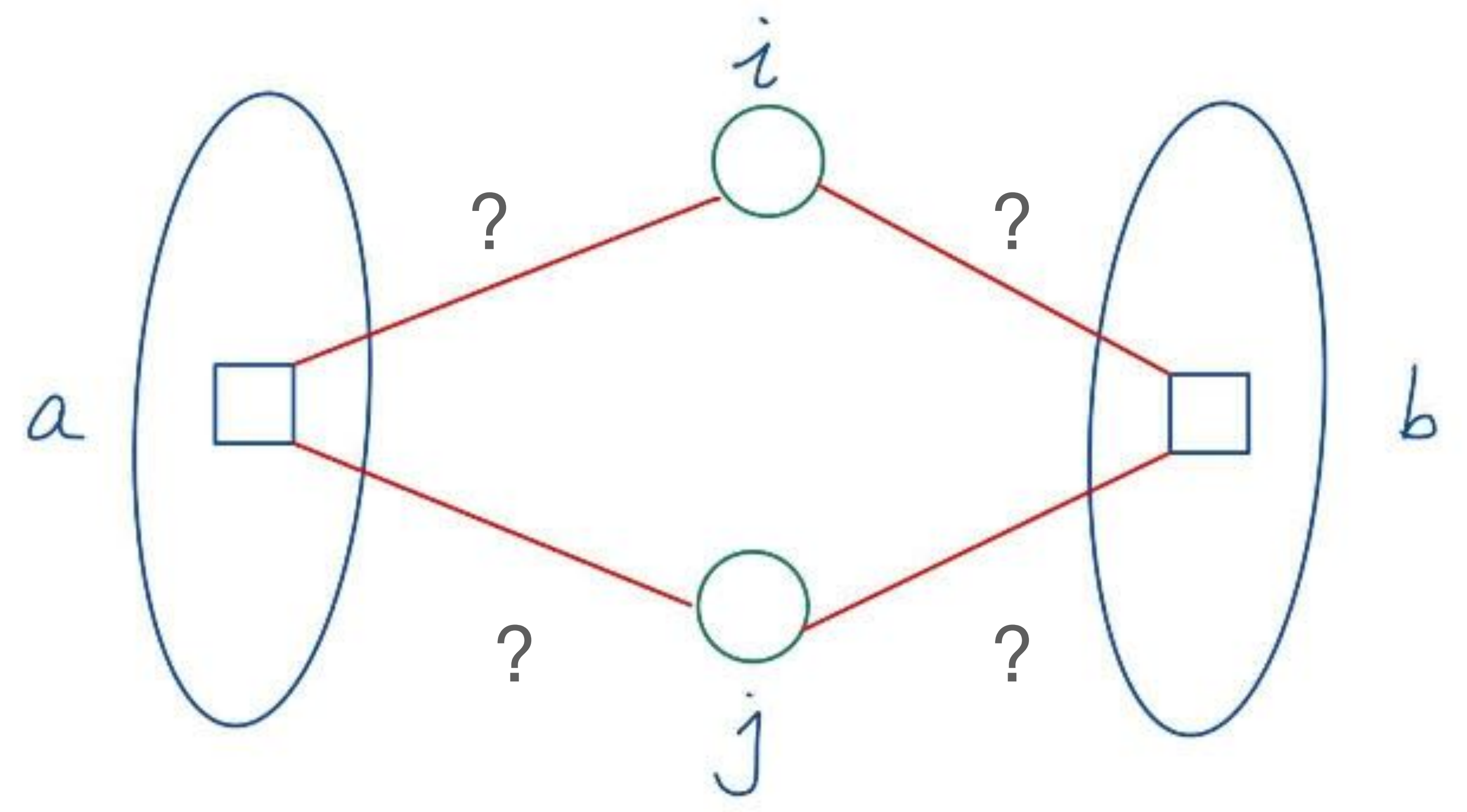
- **Exponentially** many more matrices that come up —> develop a **systematic** analysis



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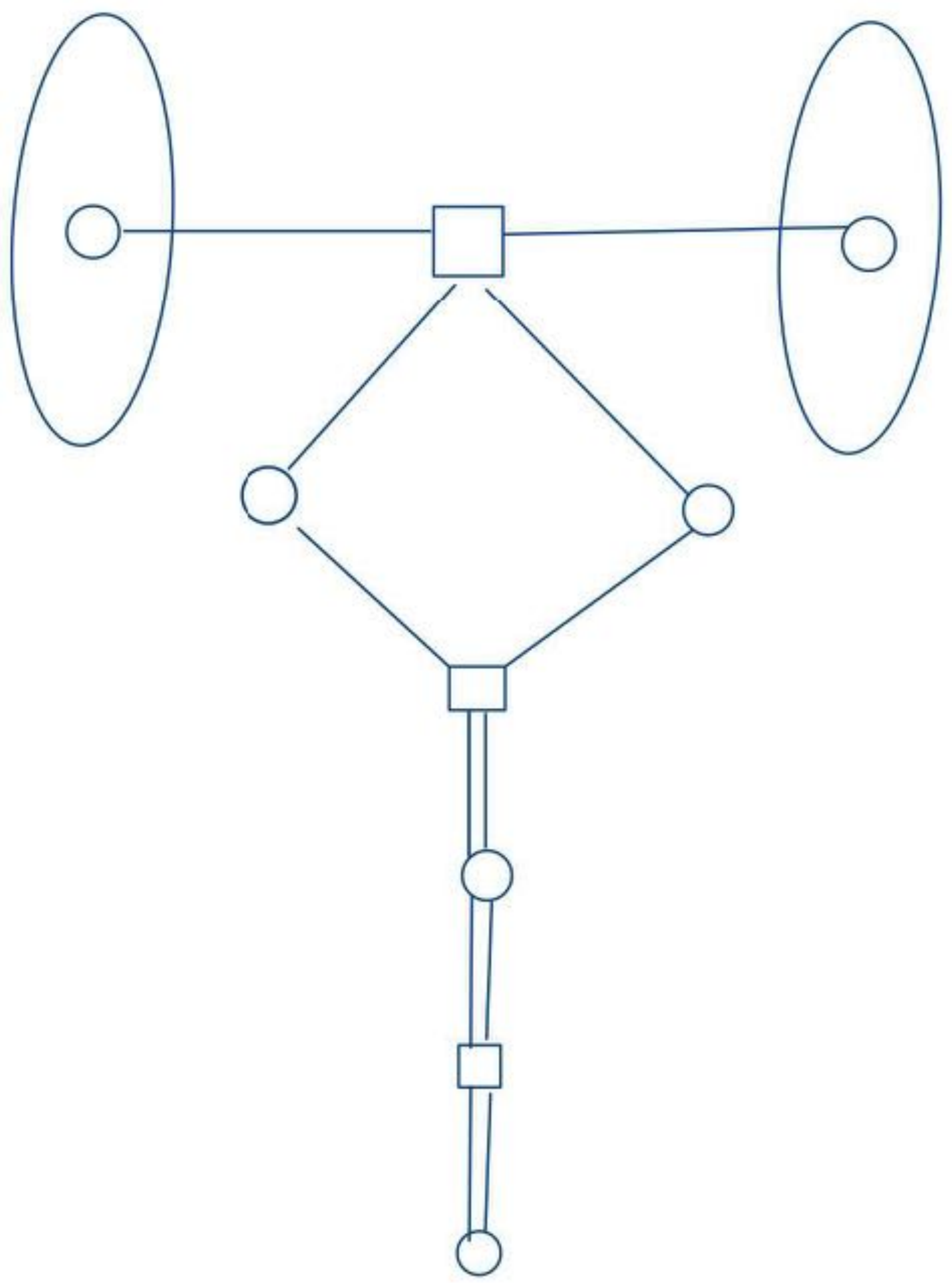

Wrapping up

1. Ellipsoid Fitting Conjecture
2. Constructing an Ellipsoid
3. Analysis via Graph Matrices
4. **A Local Machinery for Tight Norm Bounds**

Thank you!

Open Question [SCPW Conjecture] For all $\epsilon > 0$, and for sufficiently large d,

- **(Positive)** If $m \leq (1-\epsilon)\frac{d^2}{4}$, there exists such an ellipsoid w.h.p.
    - Our construction experimentally fails [PTVW '22]
- **(Negative)** If $m \geq (1+\epsilon)\frac{d^2}{4}$, there does not exist such an ellipsoid w.h.p.

References

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[KD22] Daniel M Kane and Ilias Diakonikolas. *A Nearly Tight Bound for Fitting an Ellipsoid to Gaussian Random Points*. arXiv preprint arXiv:2212.11221, 2022.

[PTVW22] Aaron Potechin, Paxton Turner, Prayaag Venkat, and Alexander S Wein. *Near-optimal fitting of ellipsoids to random points*. arXiv preprint arXiv:2208.09493, 2022.

[SCPW12] James Saunderson, Venkat Chandrasekaran, Pablo A Parrilo, and Alan S Willsky. Diagonal and low-rank matrix decompositions, correlation matrices, and ellipsoid fitting. SIAM Journal on Matrix Analysis and Applications, 33(4):1395–1416, 2012.

[SPW13] James Saunderson, Pablo A Parrilo, and Alan S Willsky. Diagonal and low-rank decompositions and fitting ellipsoids to random points. In 52nd IEEE Conference on Decision and Control, pages 6031–6036. IEEE, 2013.