Approximating Max-Cut on Bounded Degree Graphs

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Joint work with

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This is optimal under the Unique-Games Conjecture!

- $0.878 + \epsilon$ approximation is NP-hard [Khot-Kindler-Mossel-O'Donnell'07].

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- Extend to weighted instances of Max-2LIN (generalization of Max-Cut).

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Note: SDP $\geq$ OPT.

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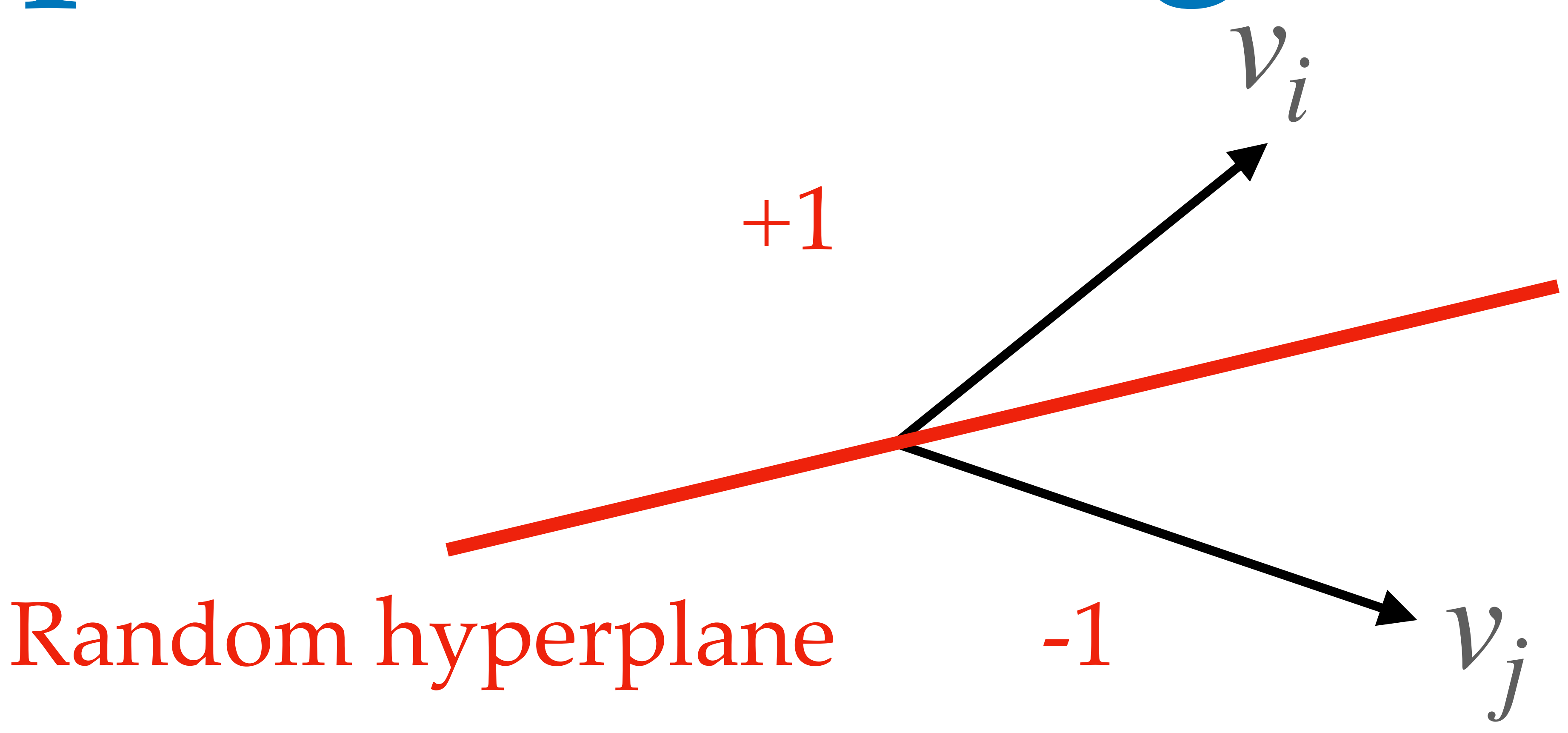

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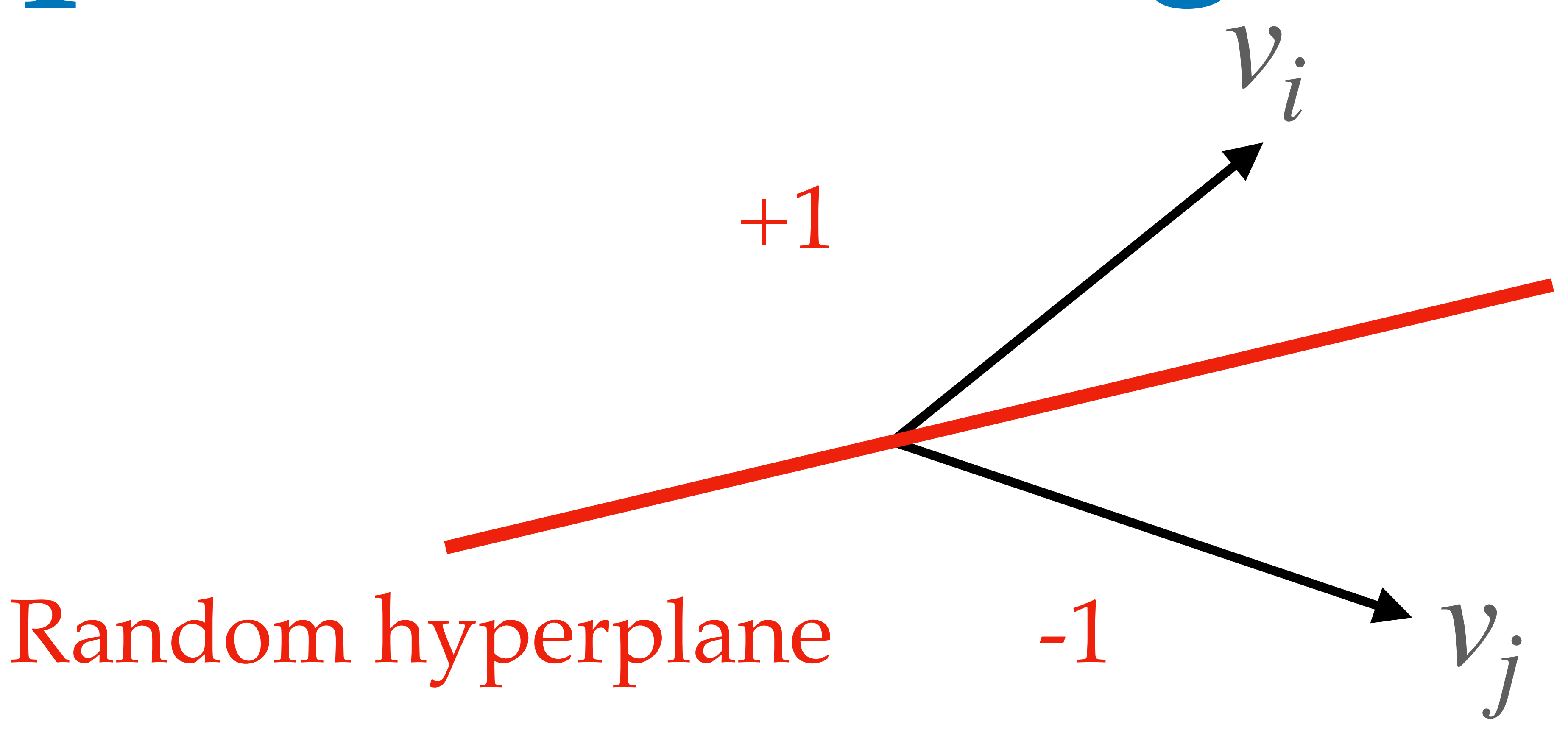

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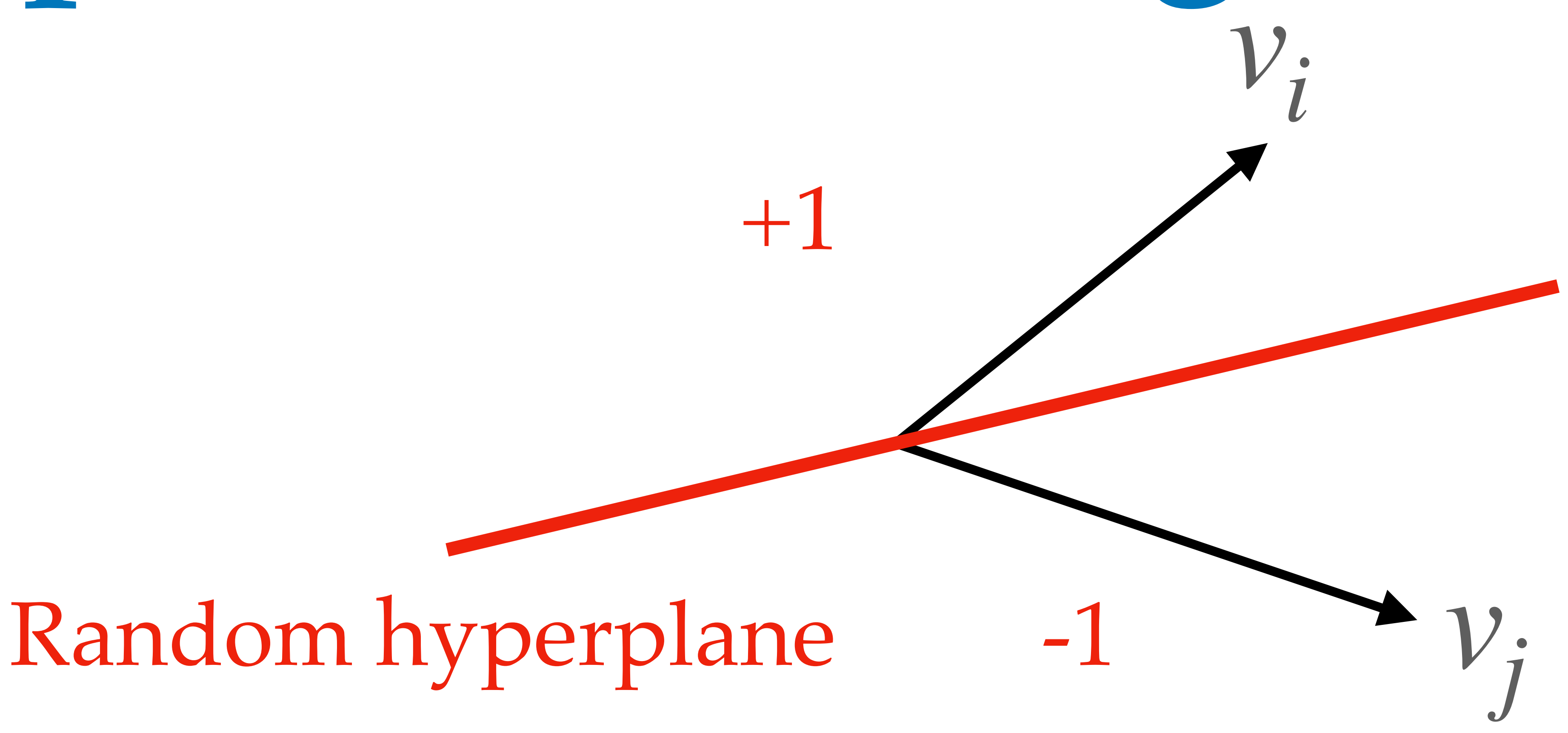

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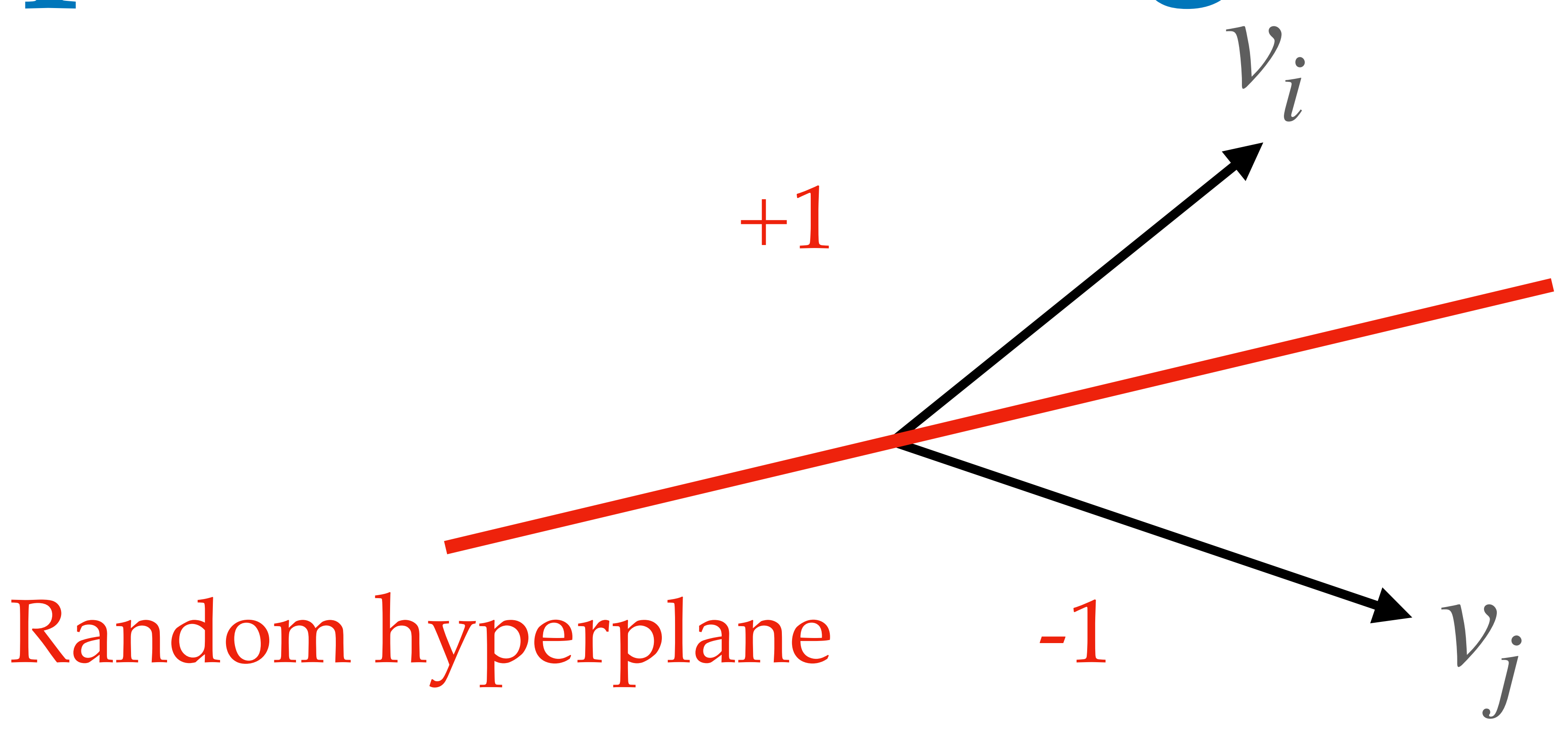

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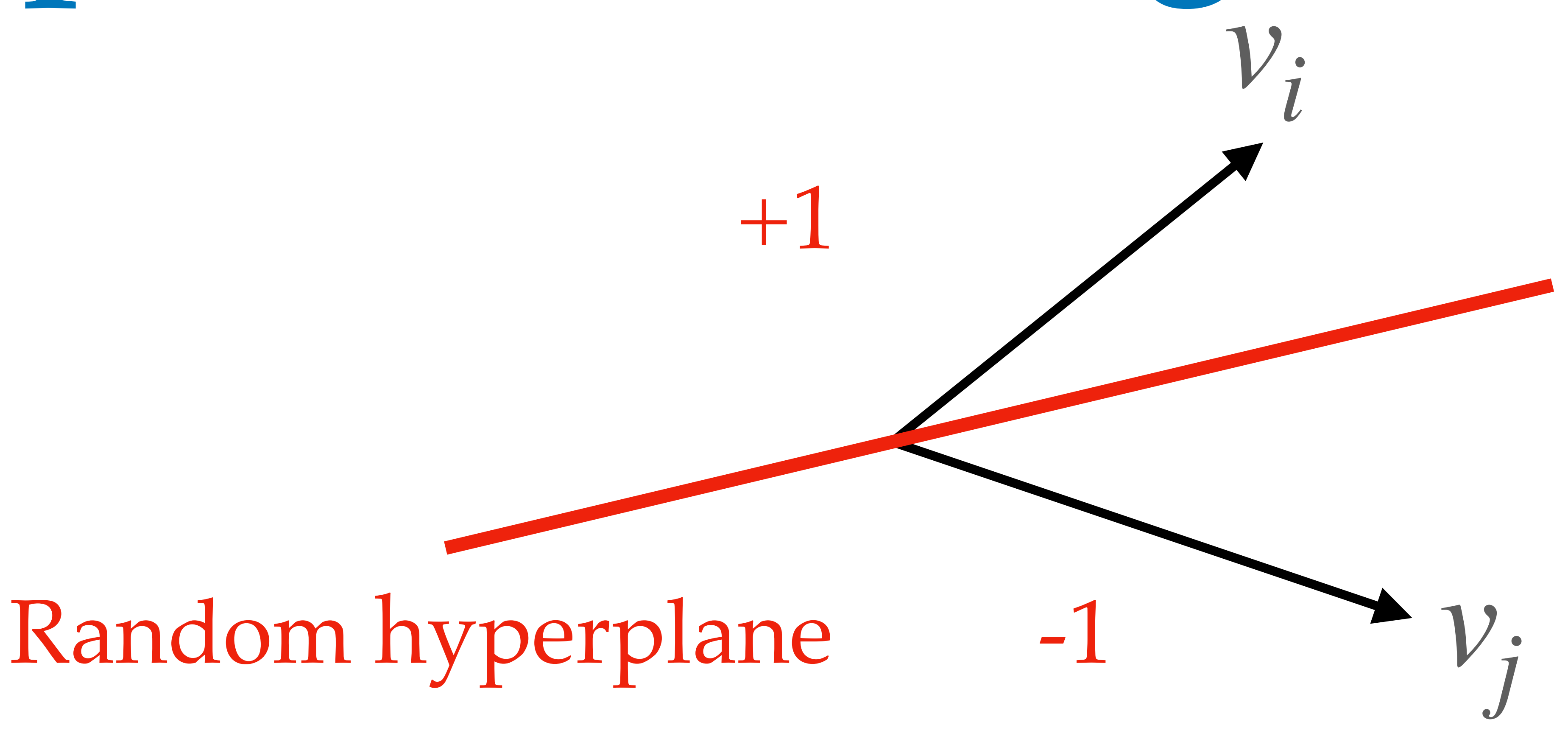

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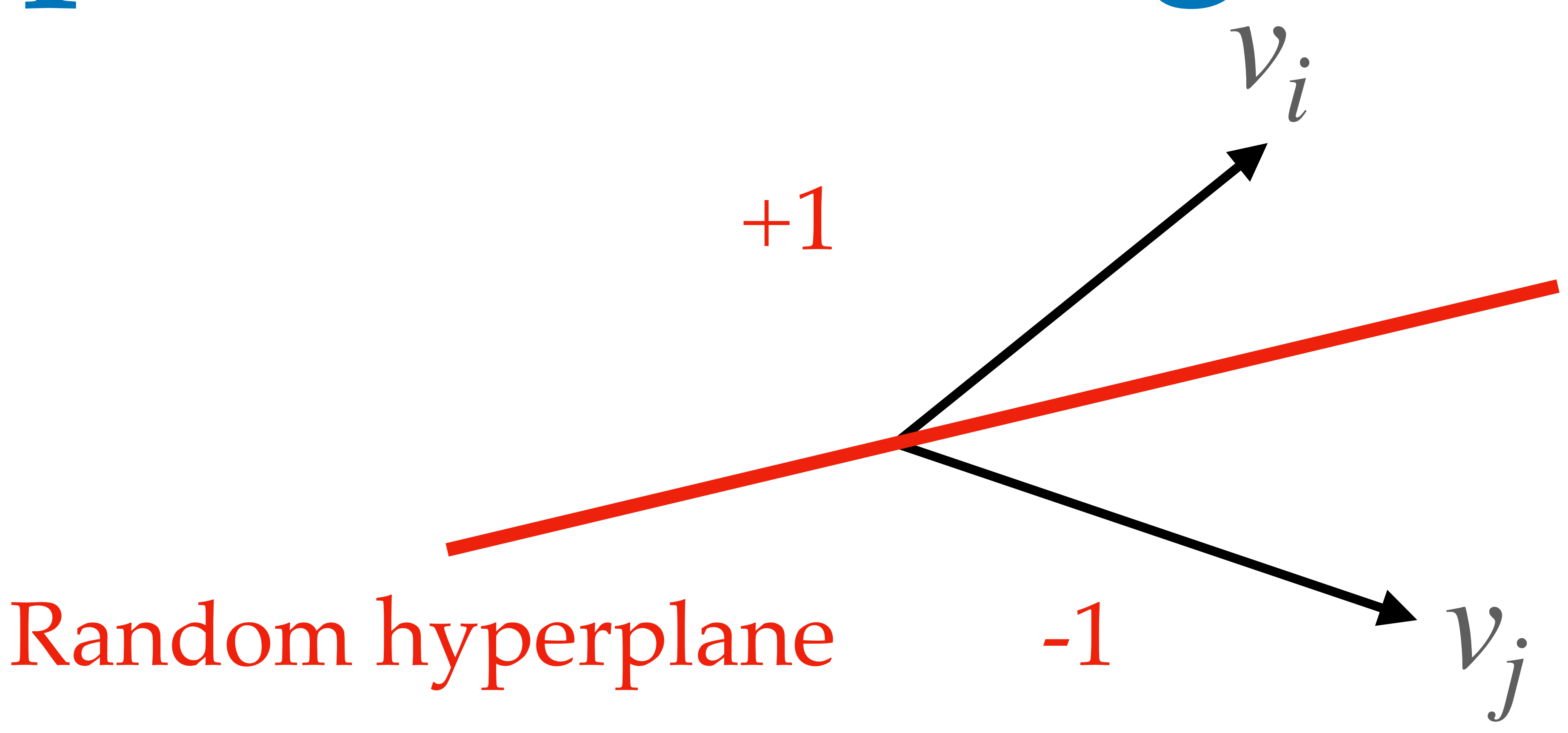

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$$\mathbb{E}[\text{Cut}] \geq 0.878 \cdot \text{SDP} \geq 0.878 \cdot \text{OPT}.$$

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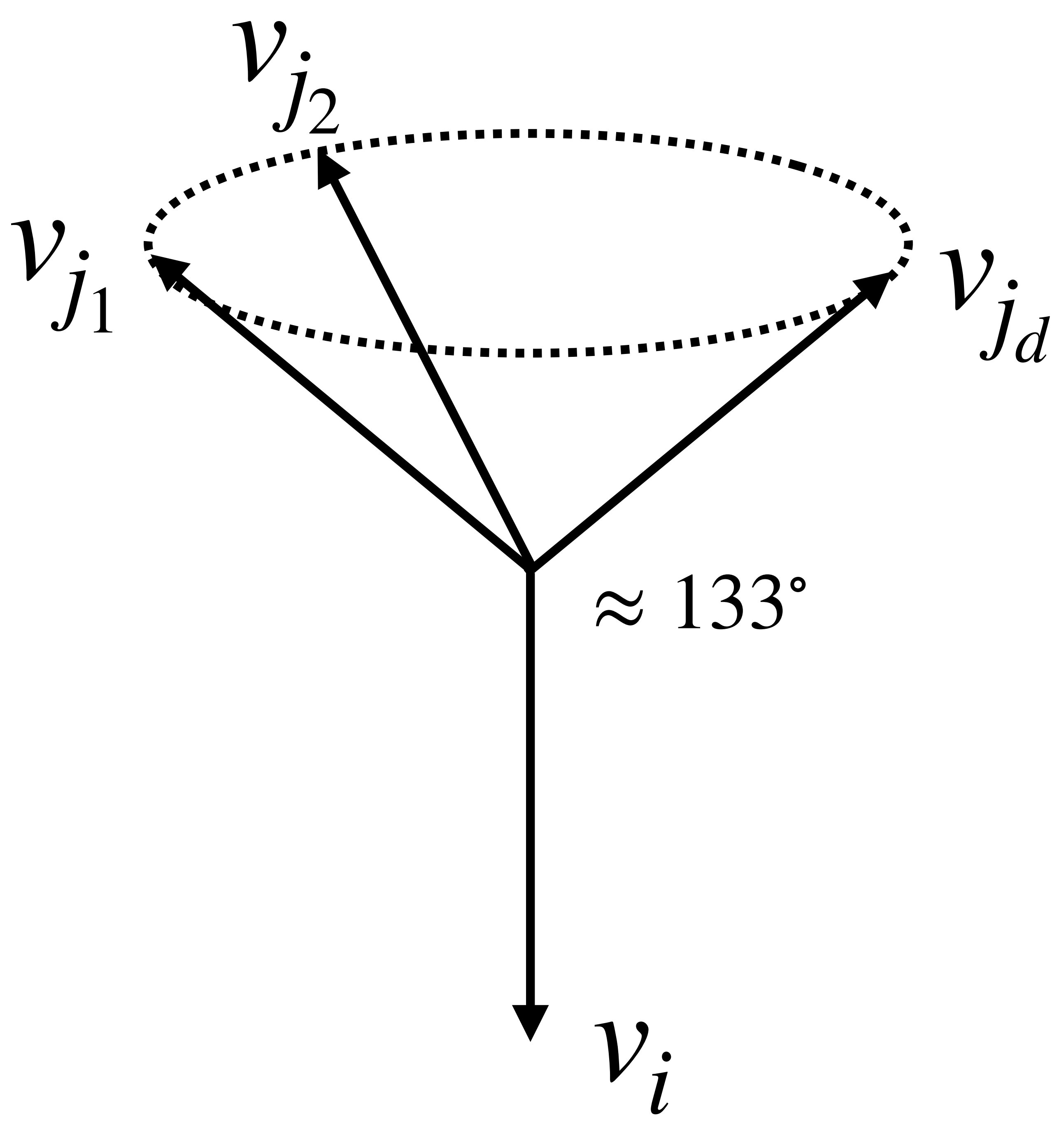

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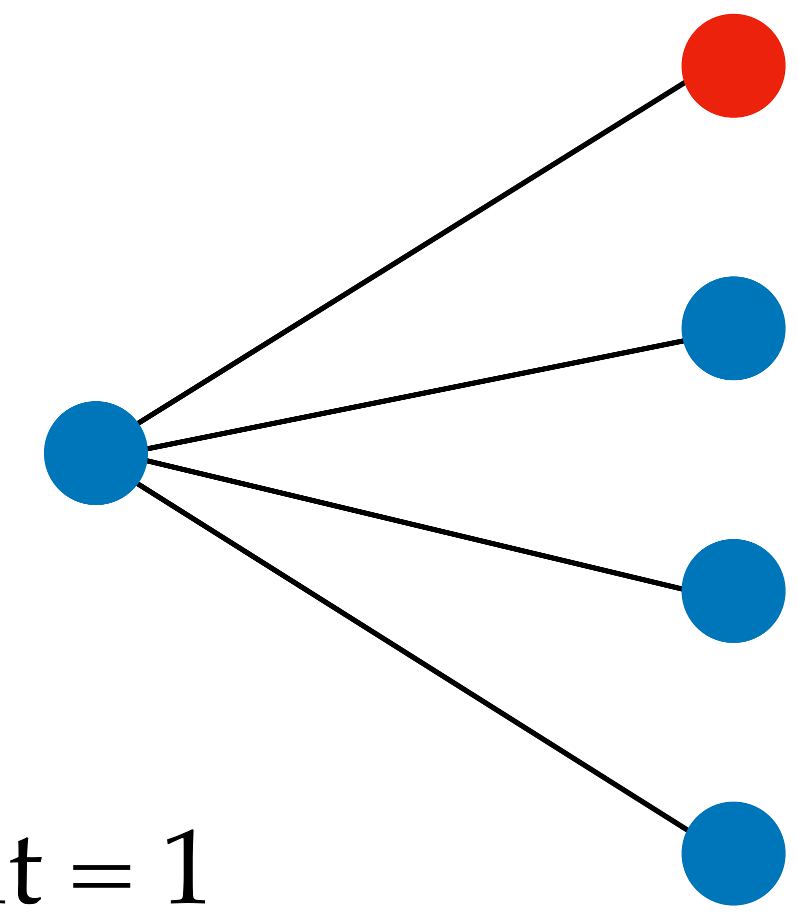

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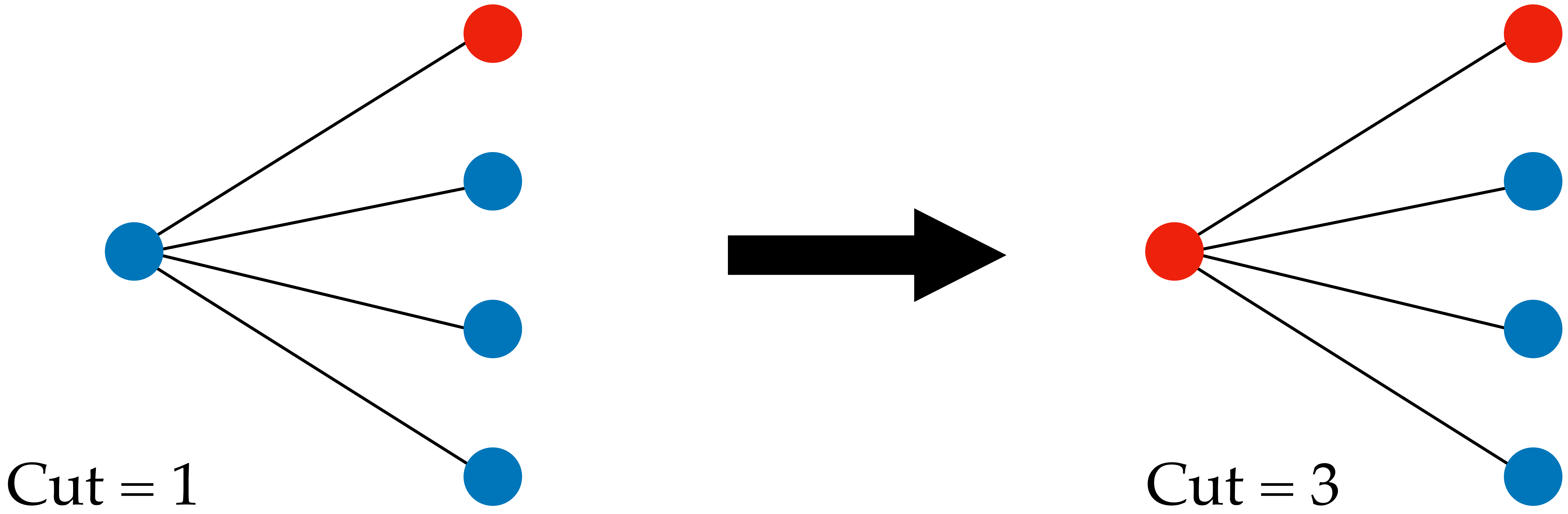

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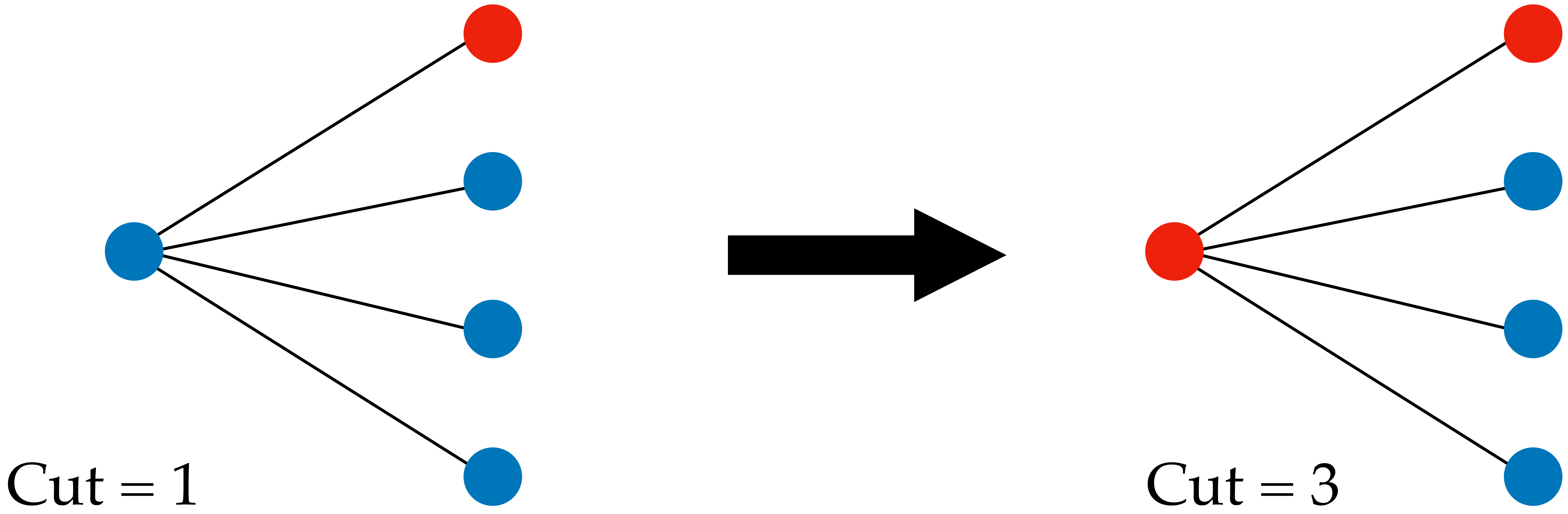

What's the probability that this occurs?

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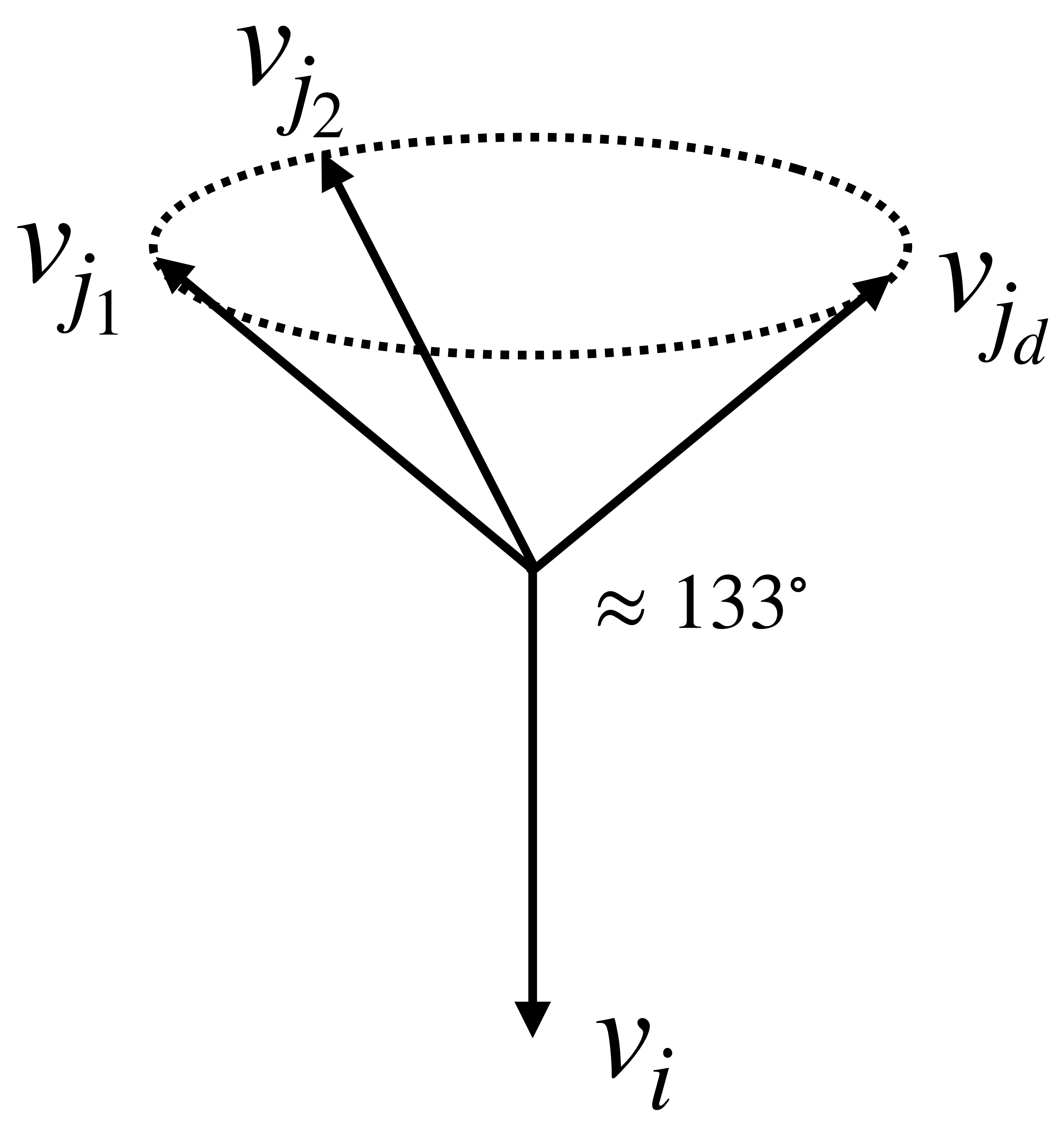

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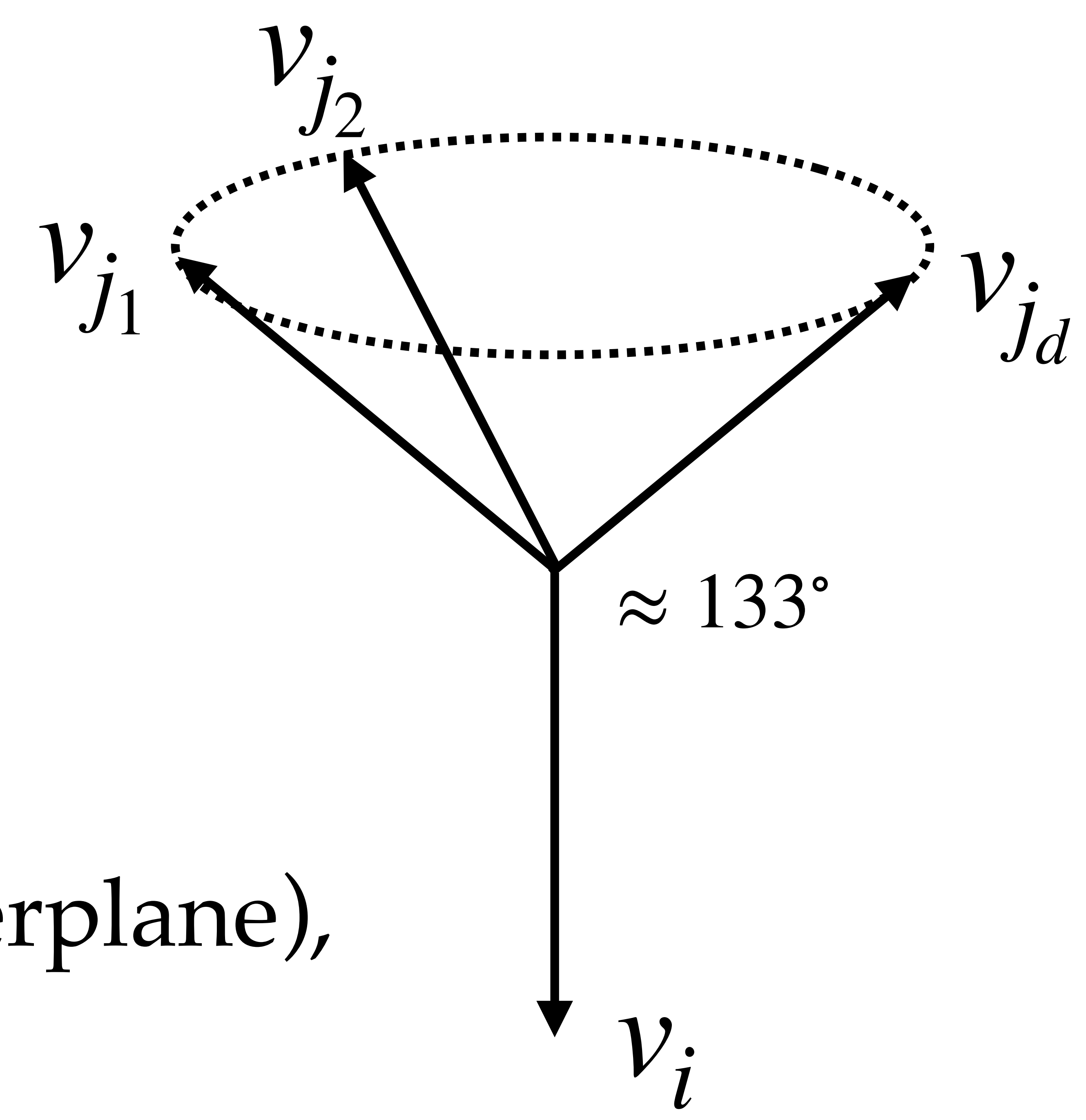

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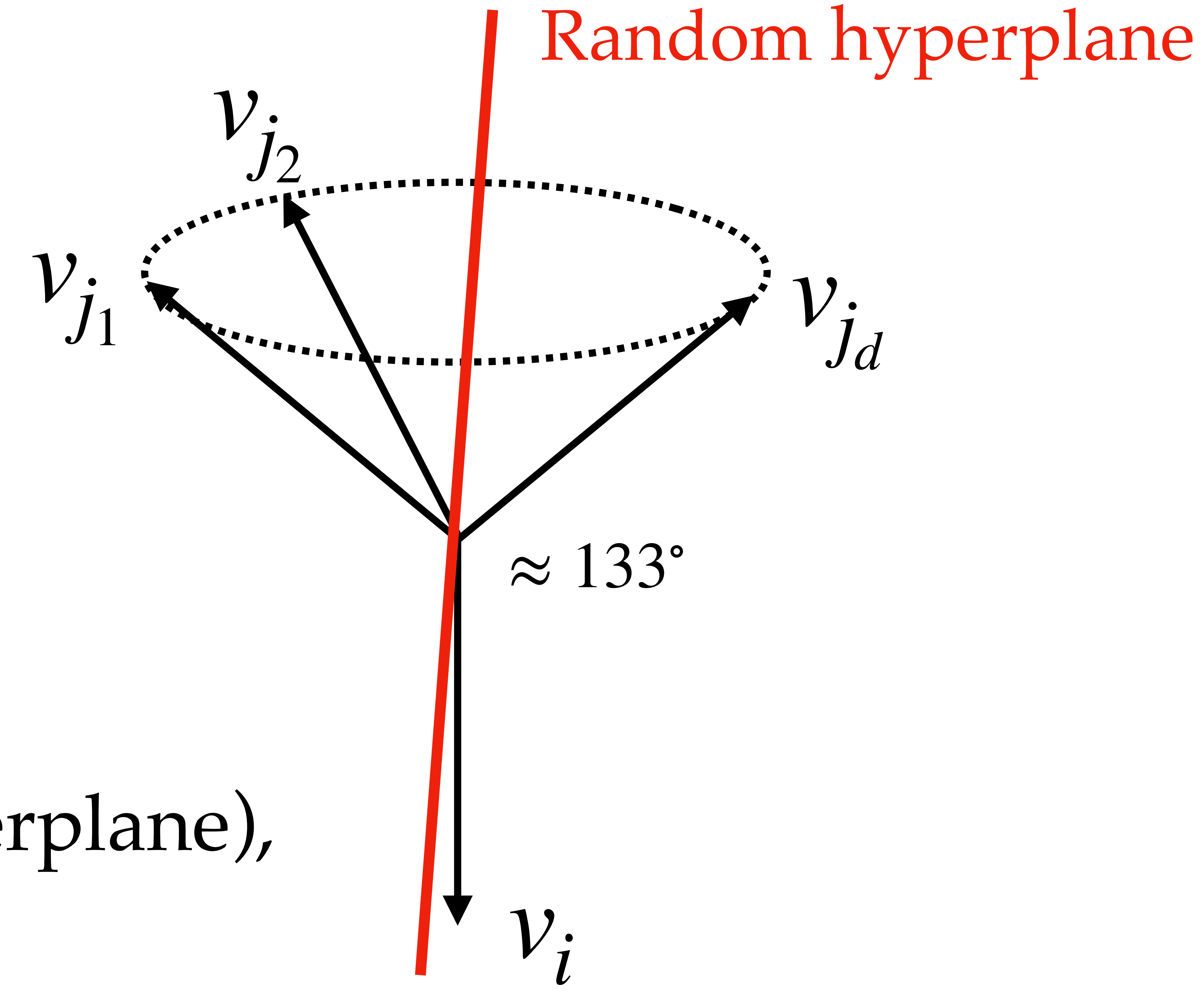

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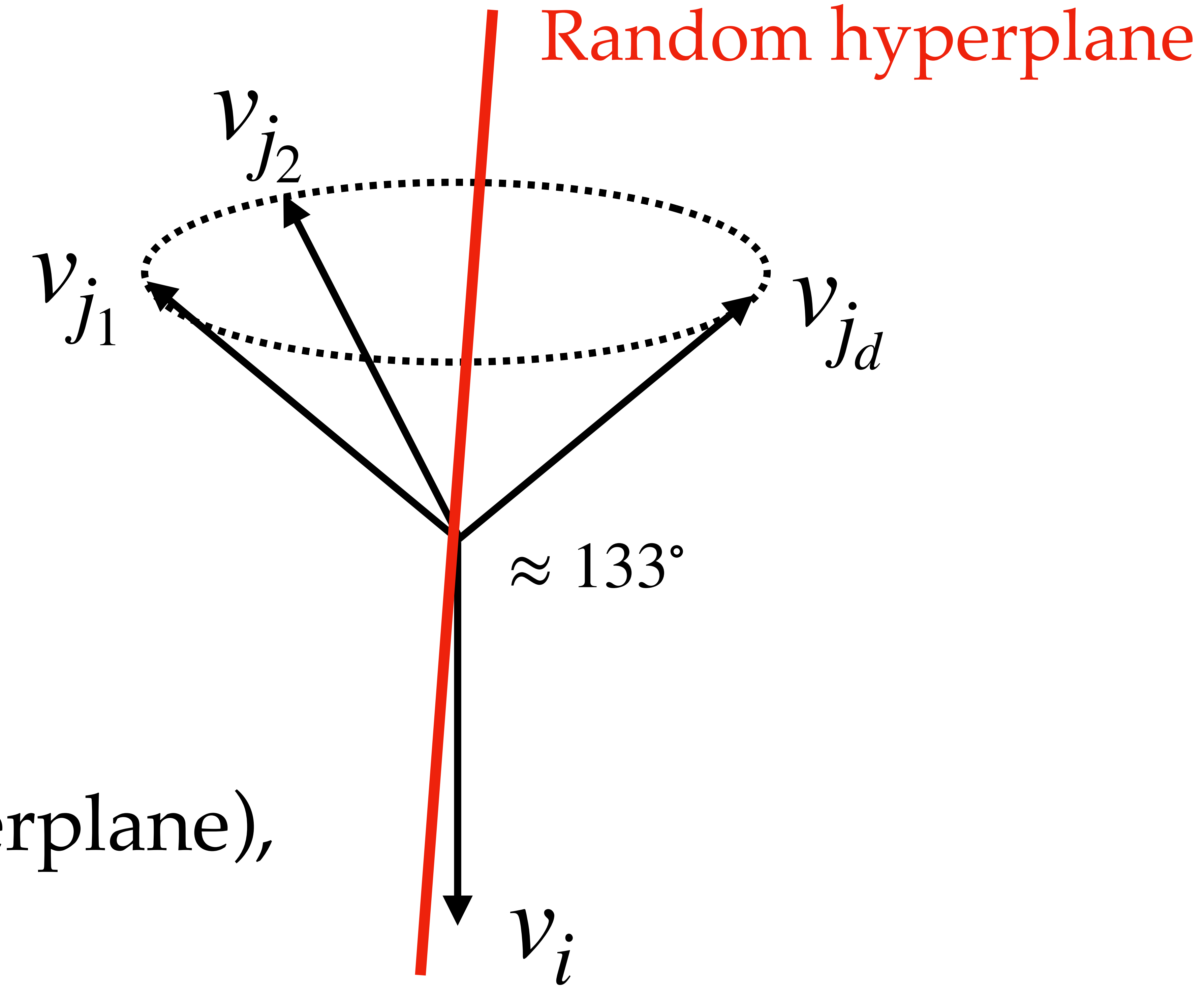

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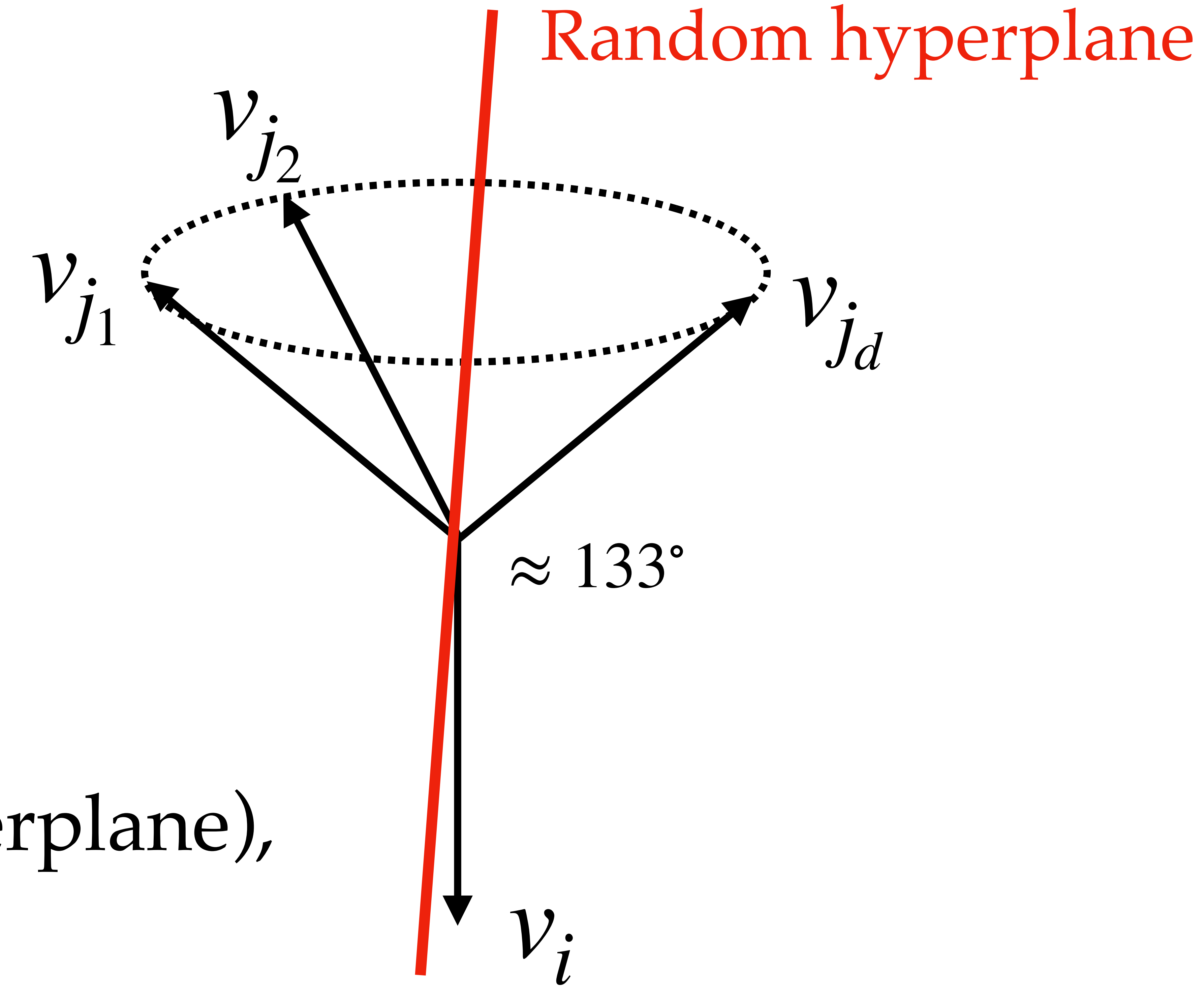

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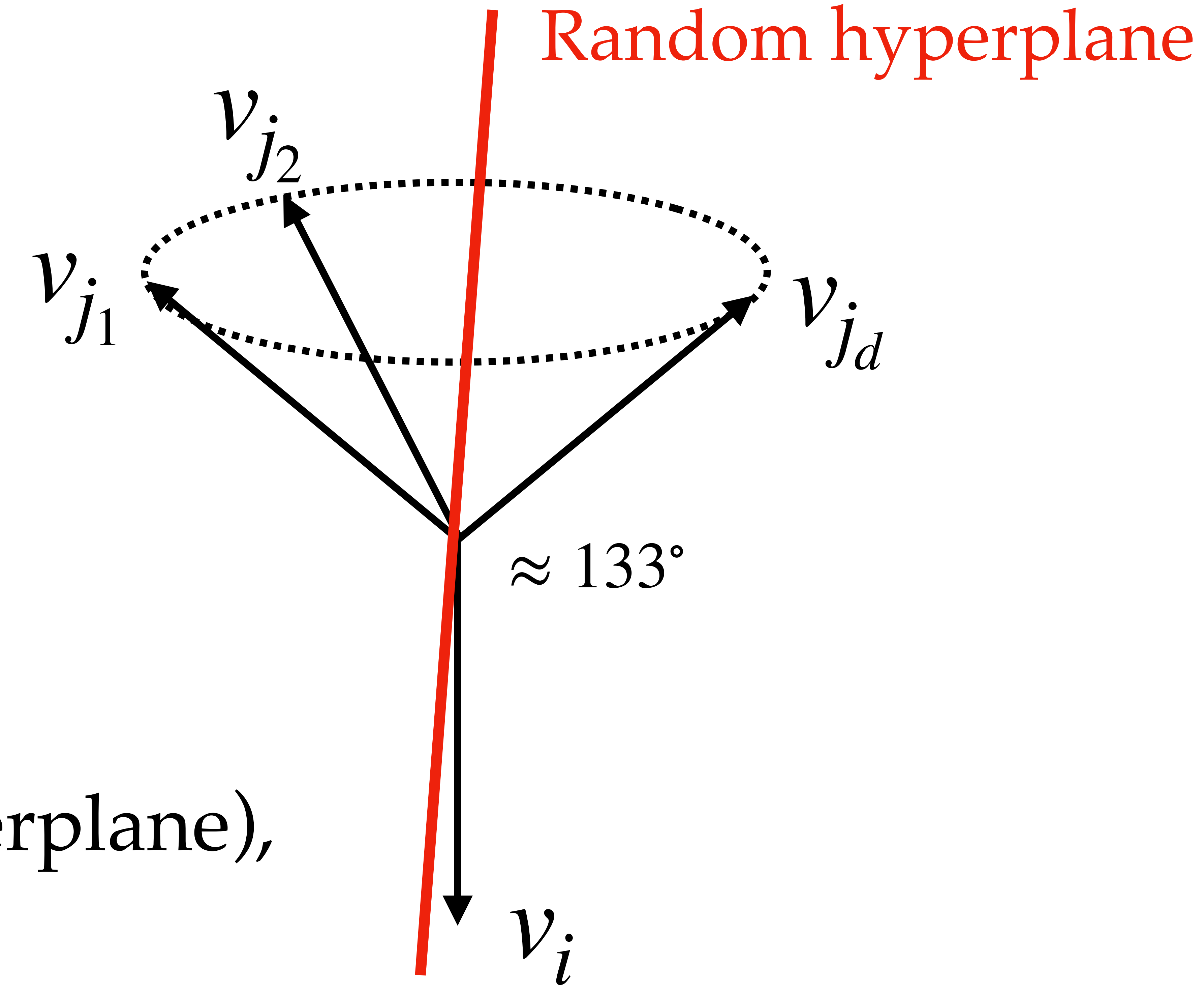

Loses an extra $1/d$ factor to handle when d is even.

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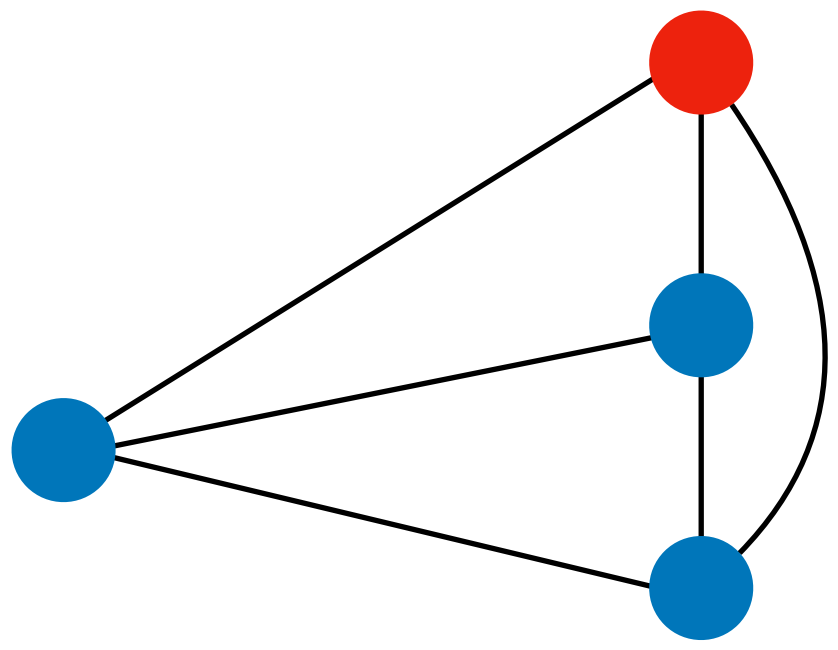

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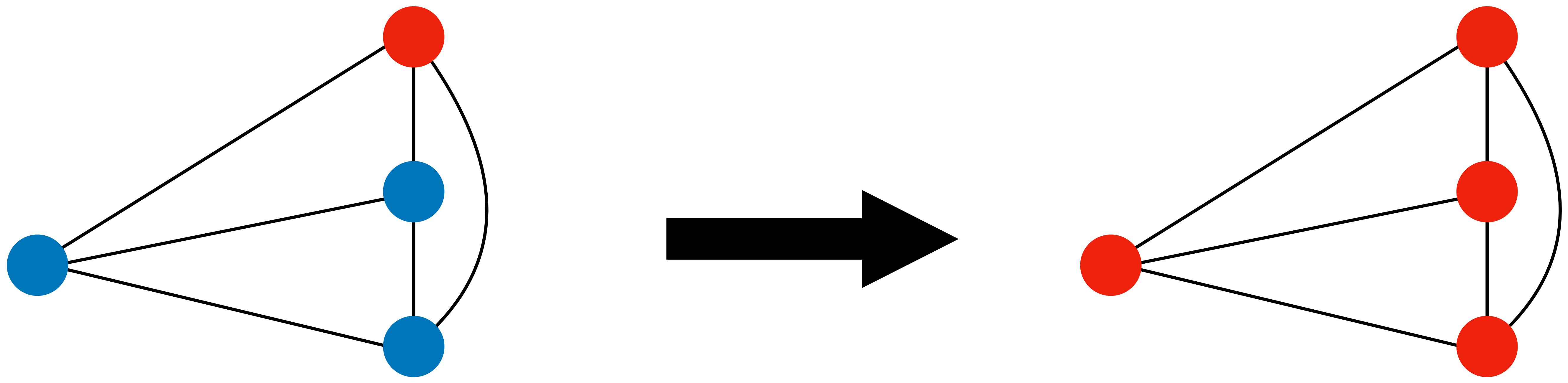

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[Florén'16] handles the dependencies better to get $\Omega(1/d^3)$ improvement.

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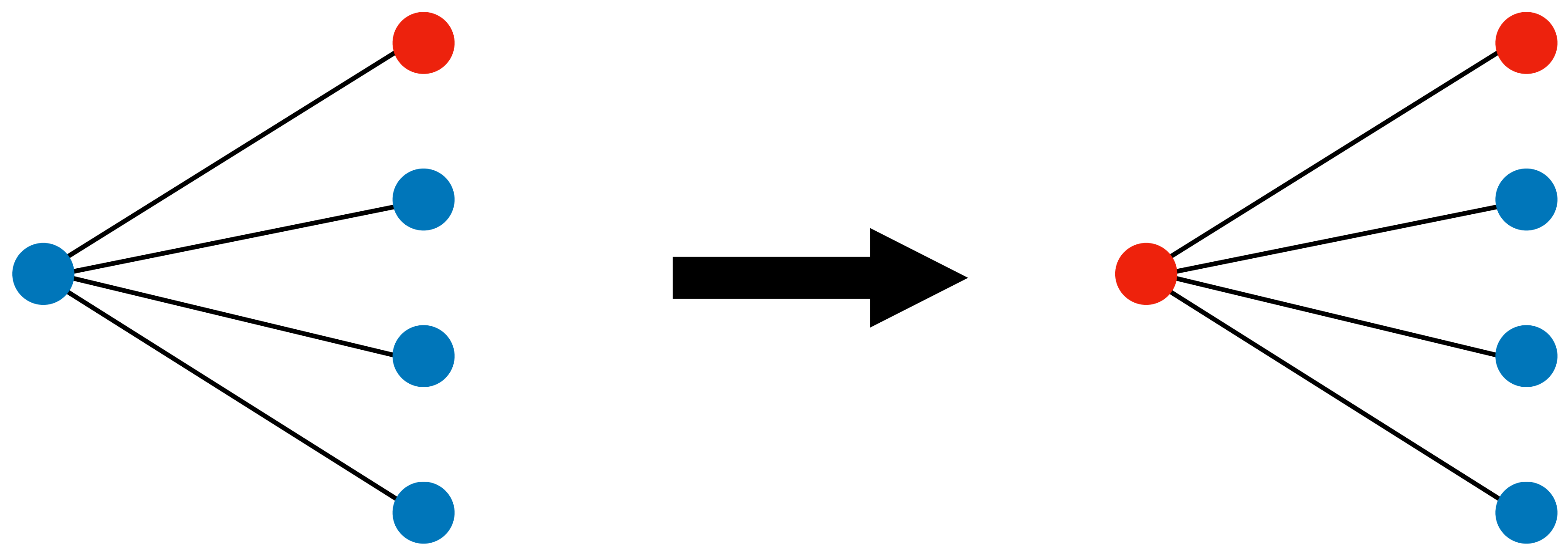

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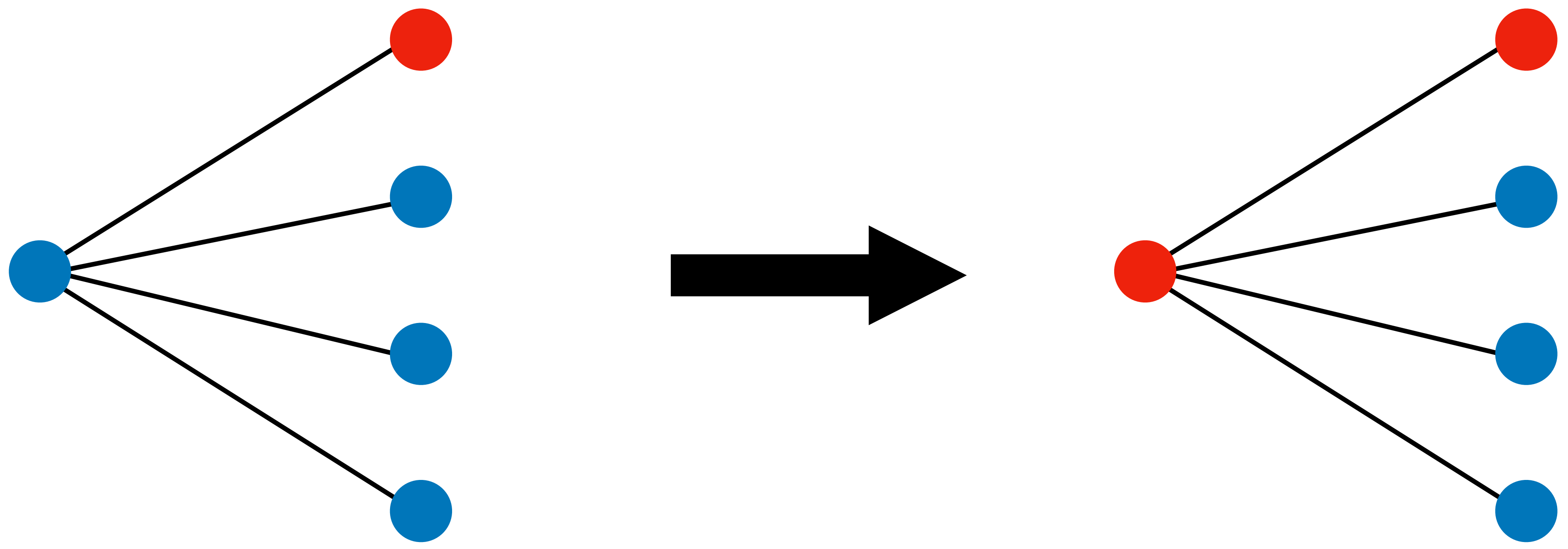

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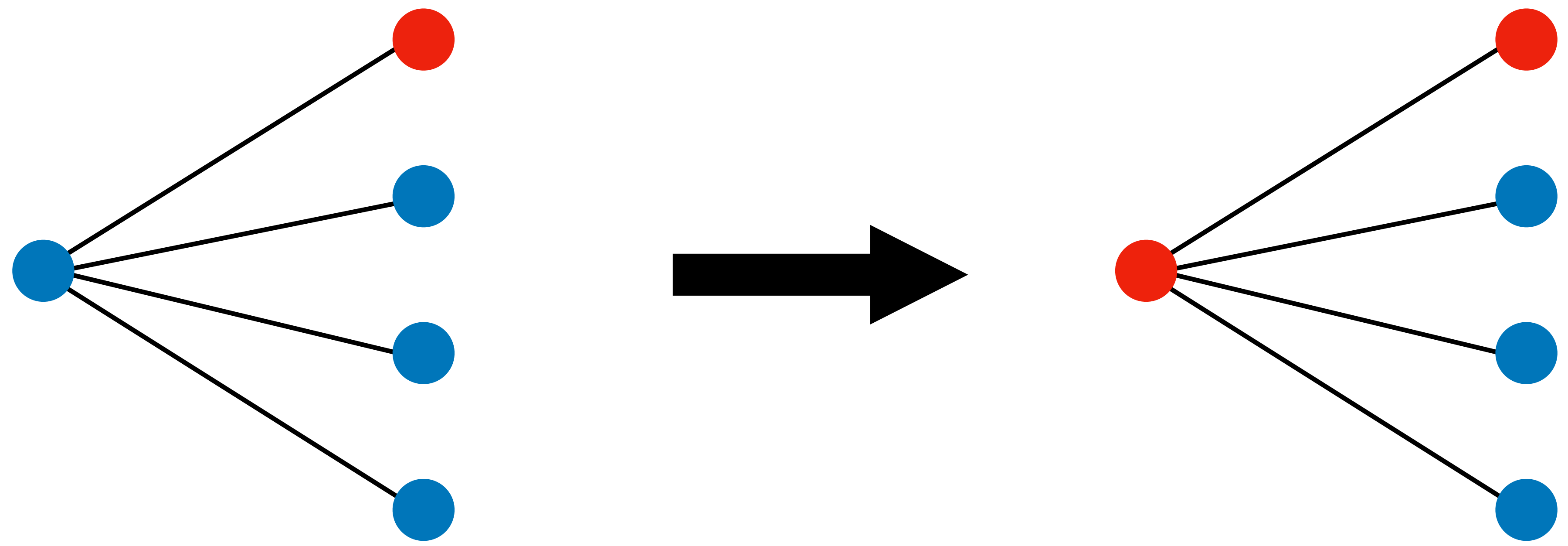

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* Don't need to handle even/odd degrees.

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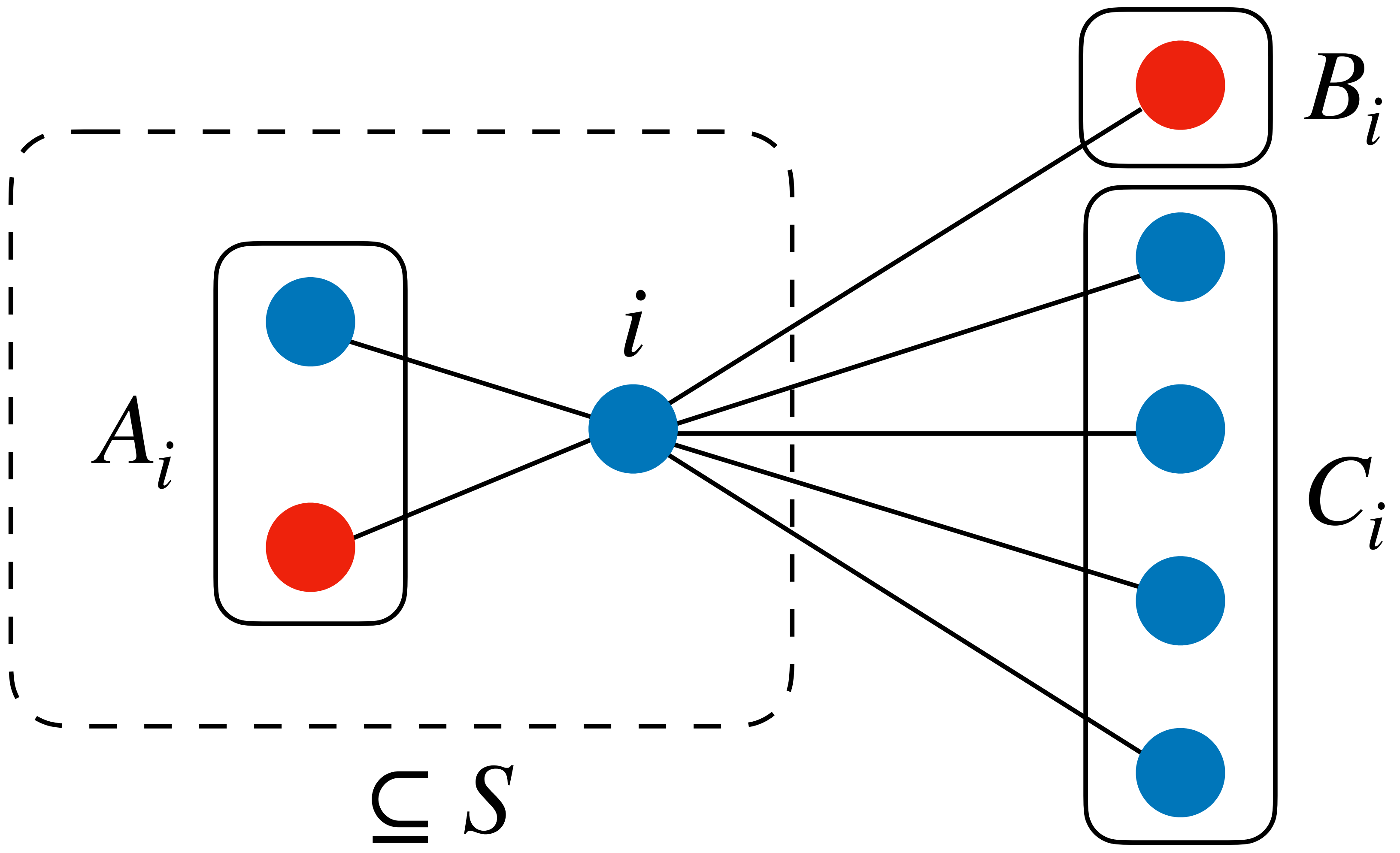

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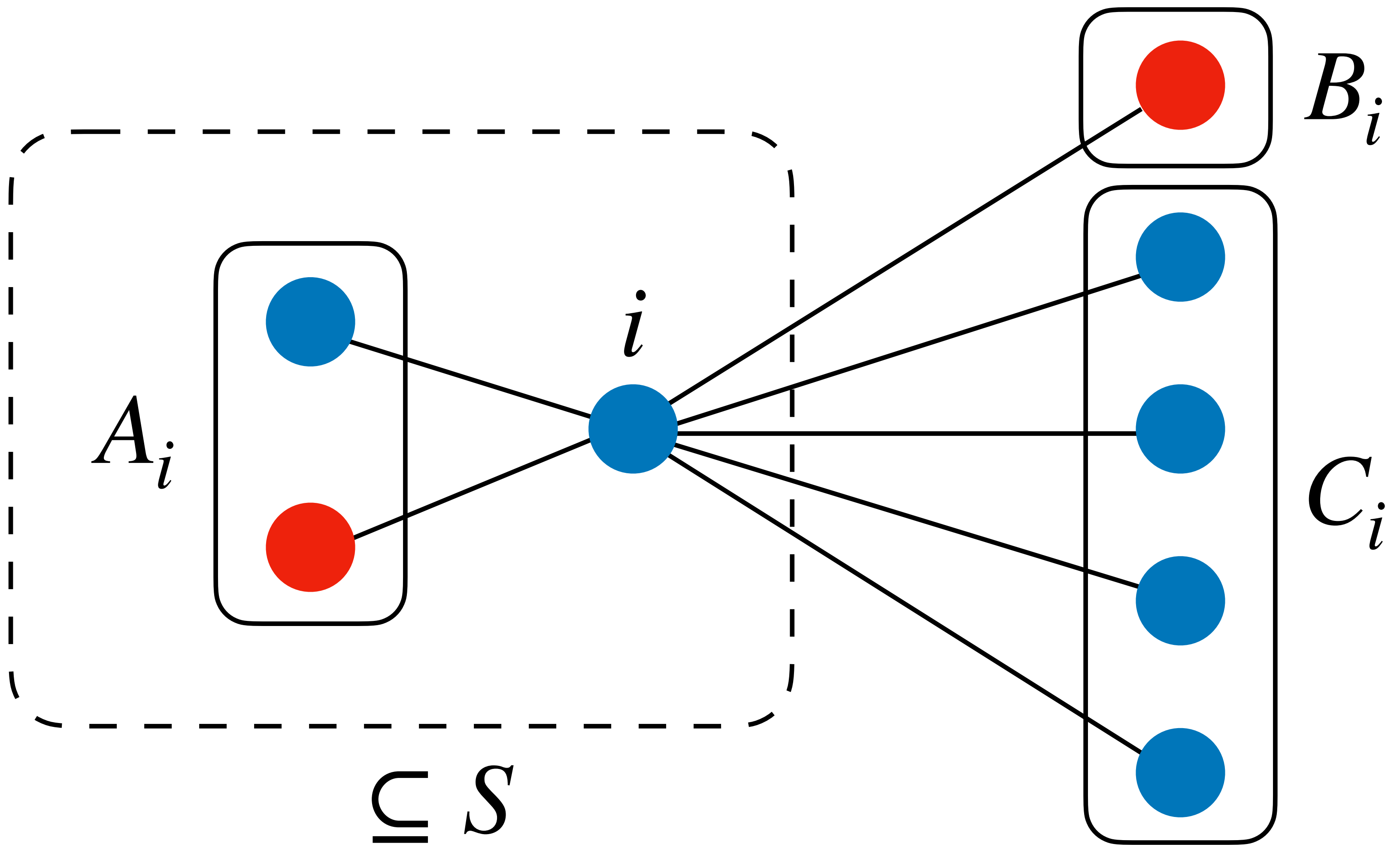

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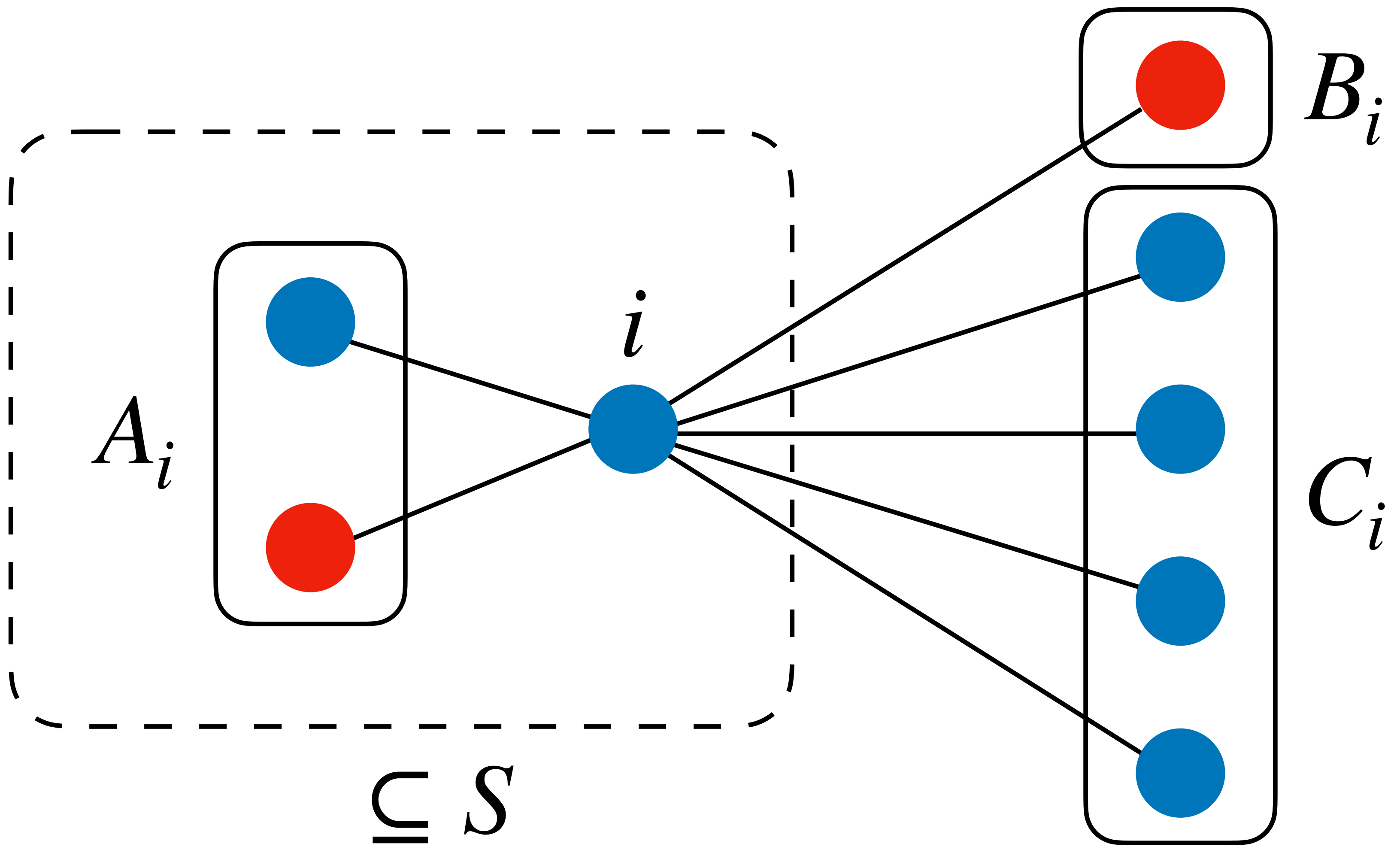

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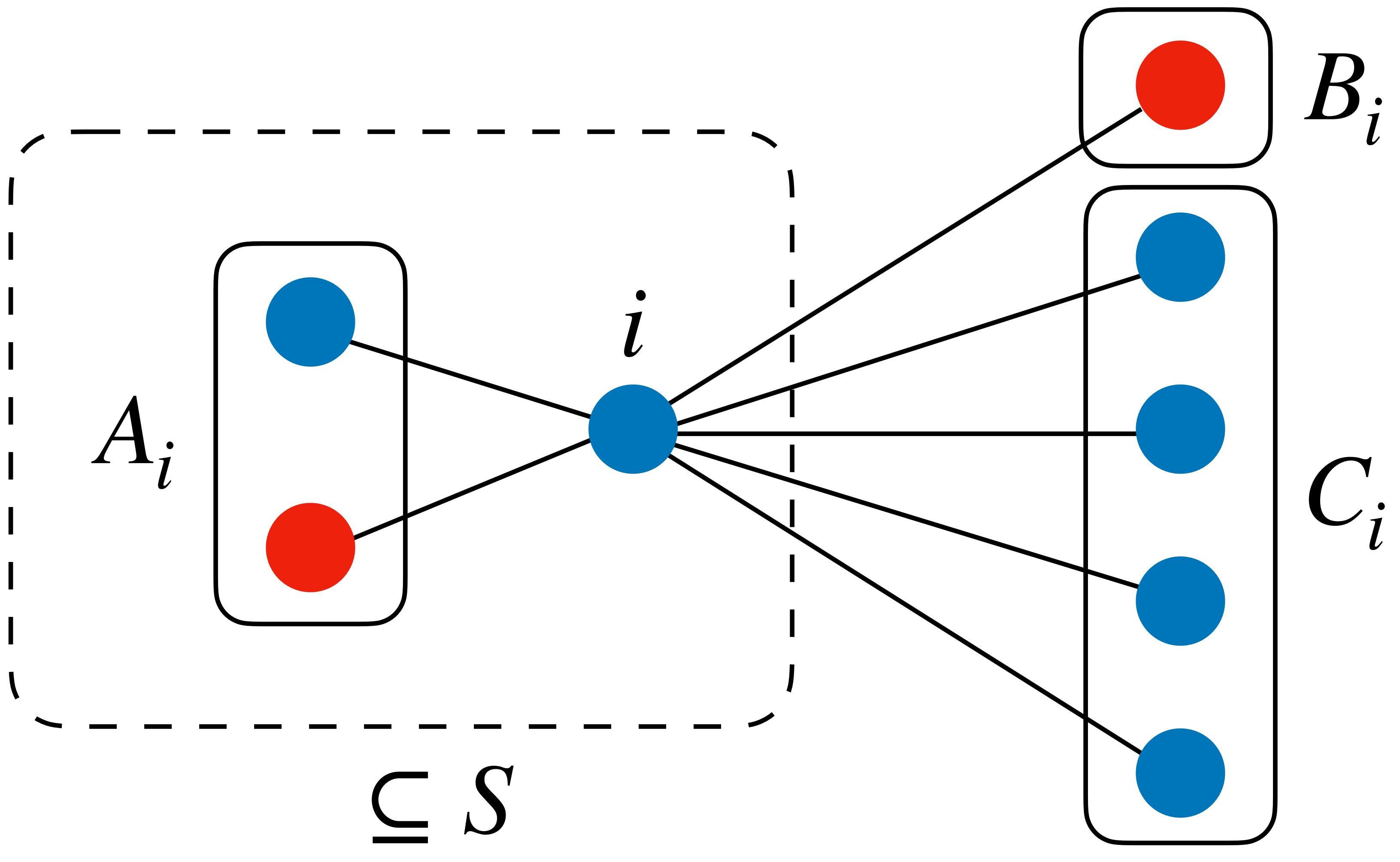

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It doesn't matter how A_i is flipped in other iterations!

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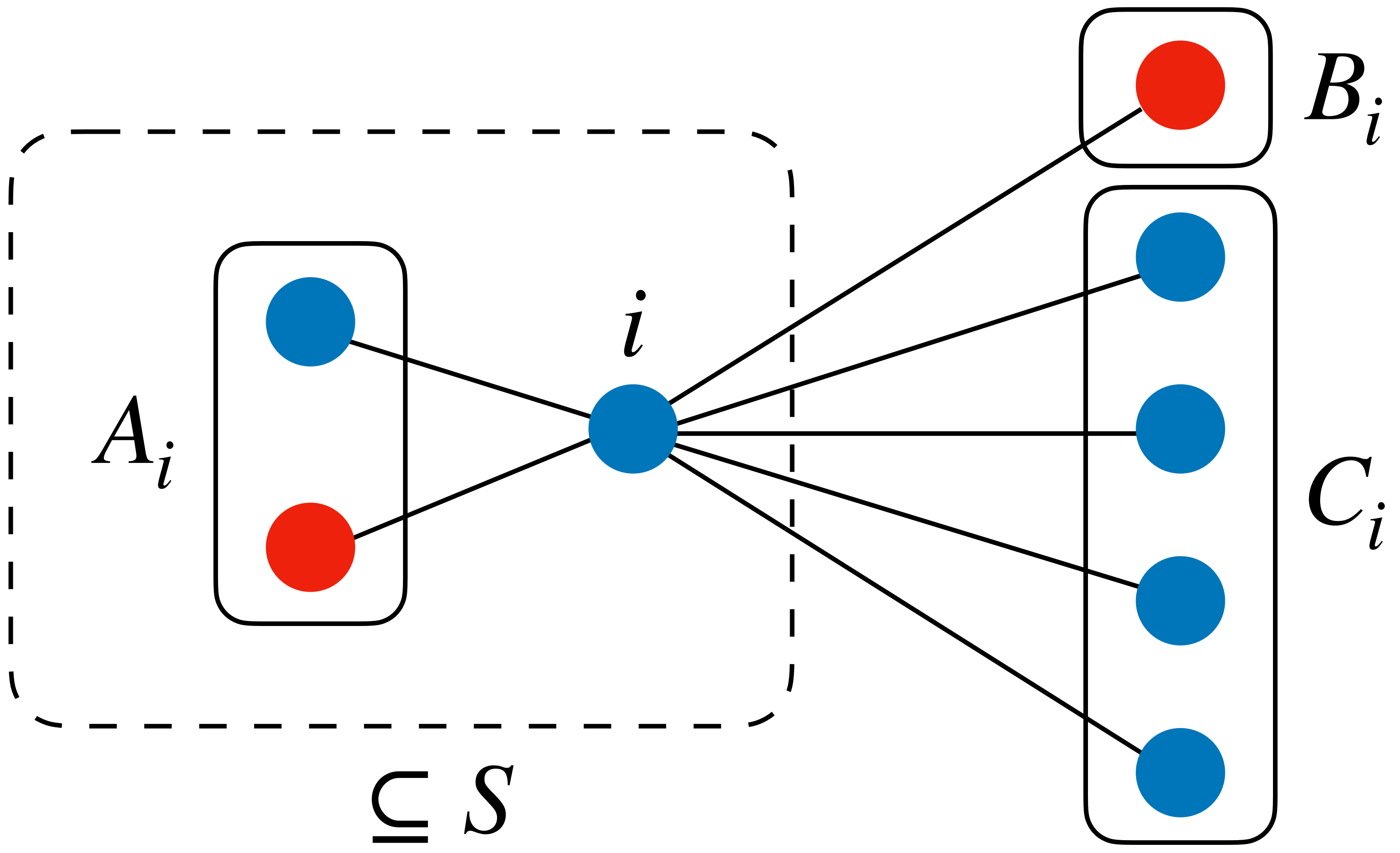

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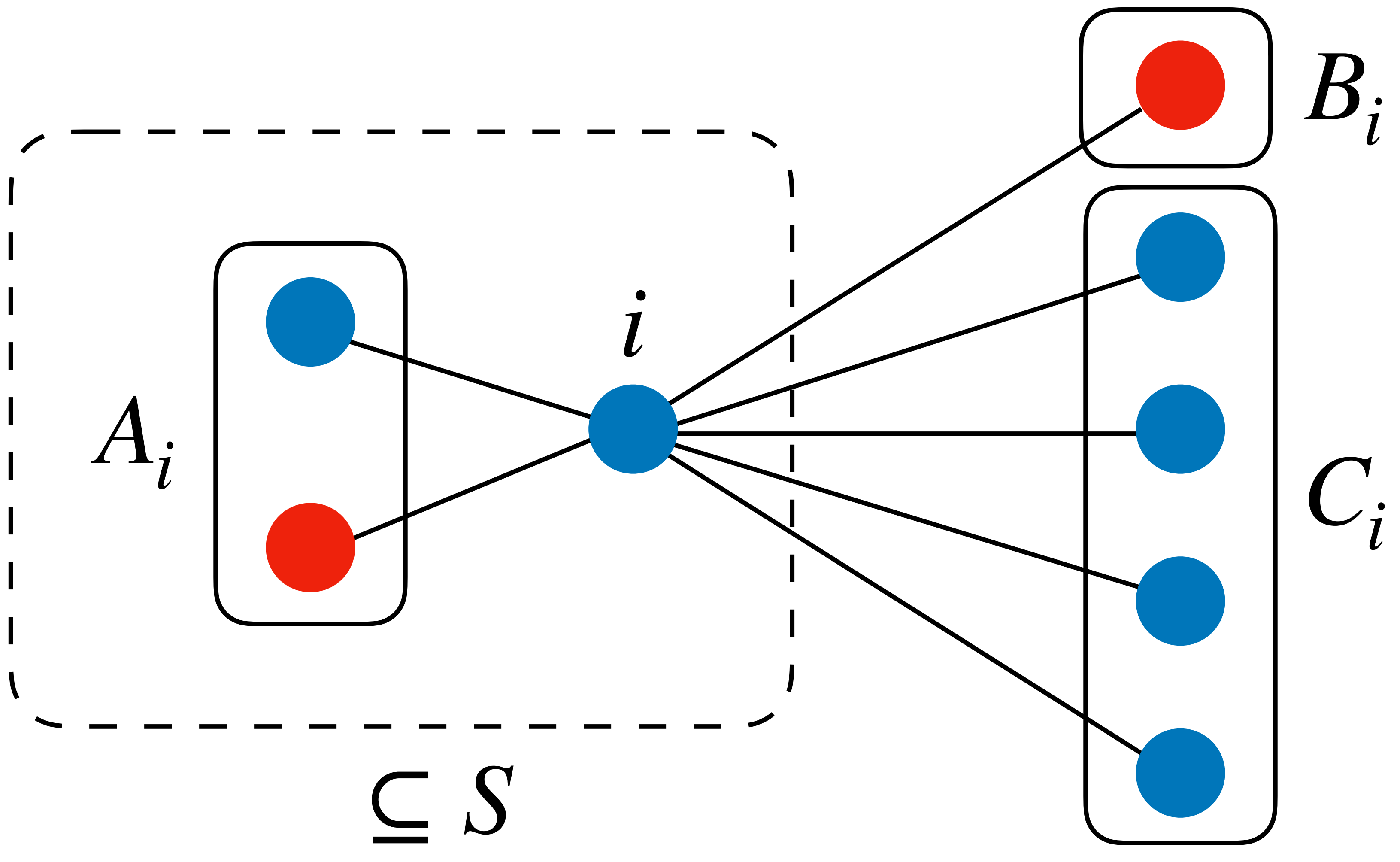

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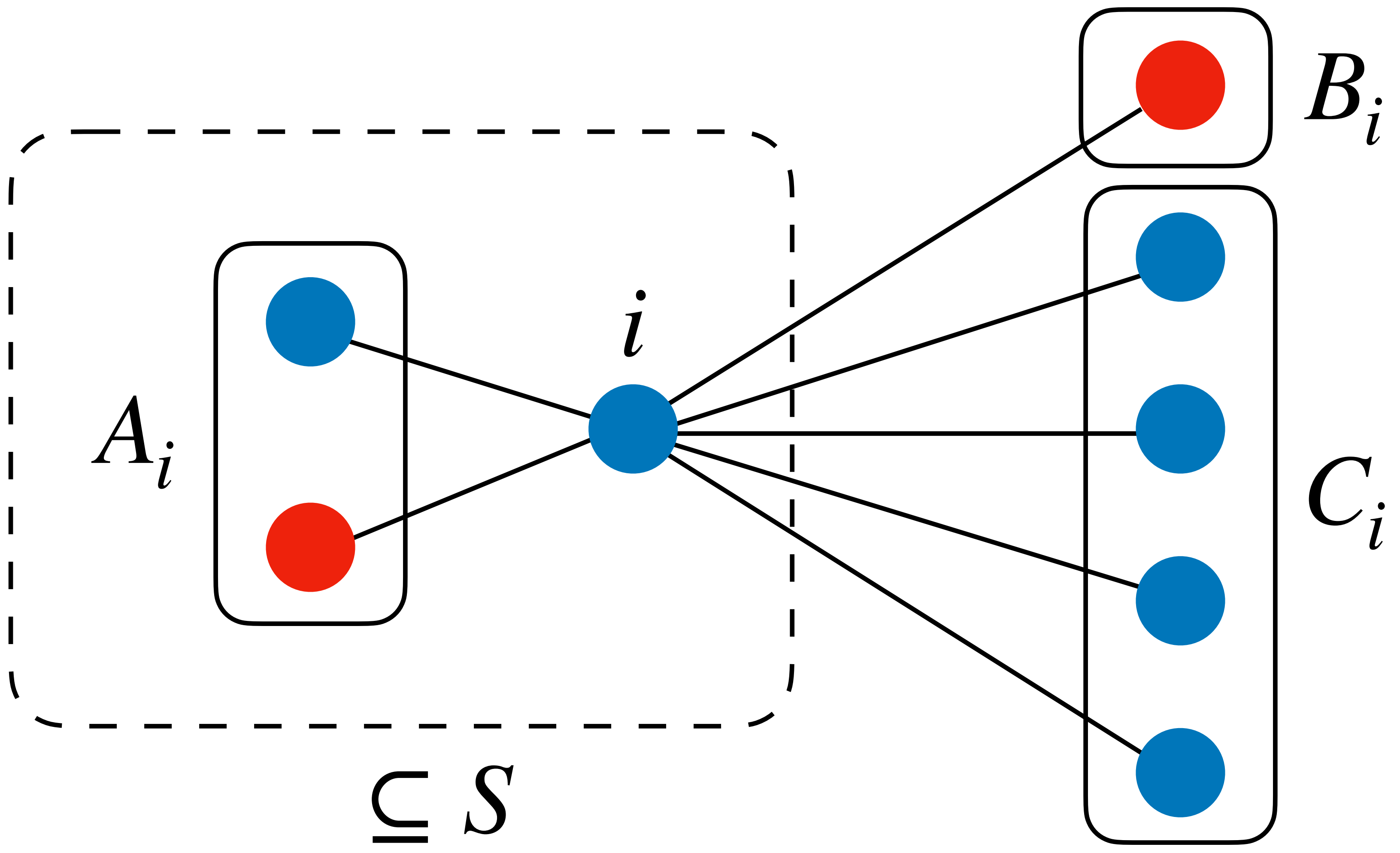
Main Lemma. $\mathbb{E}[\Delta_i \mid i \in S] \geq \tilde{\Omega}(1)$.



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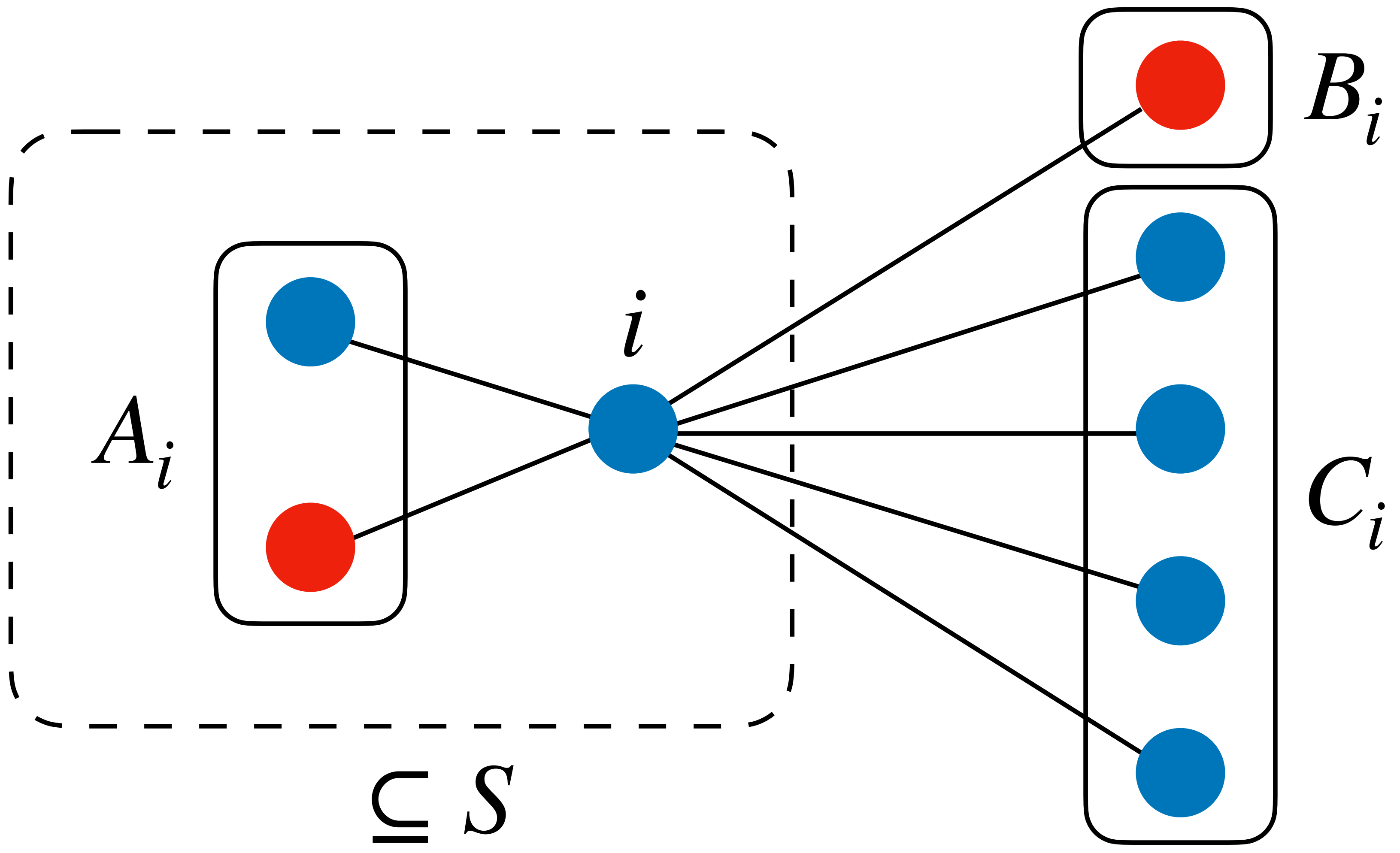

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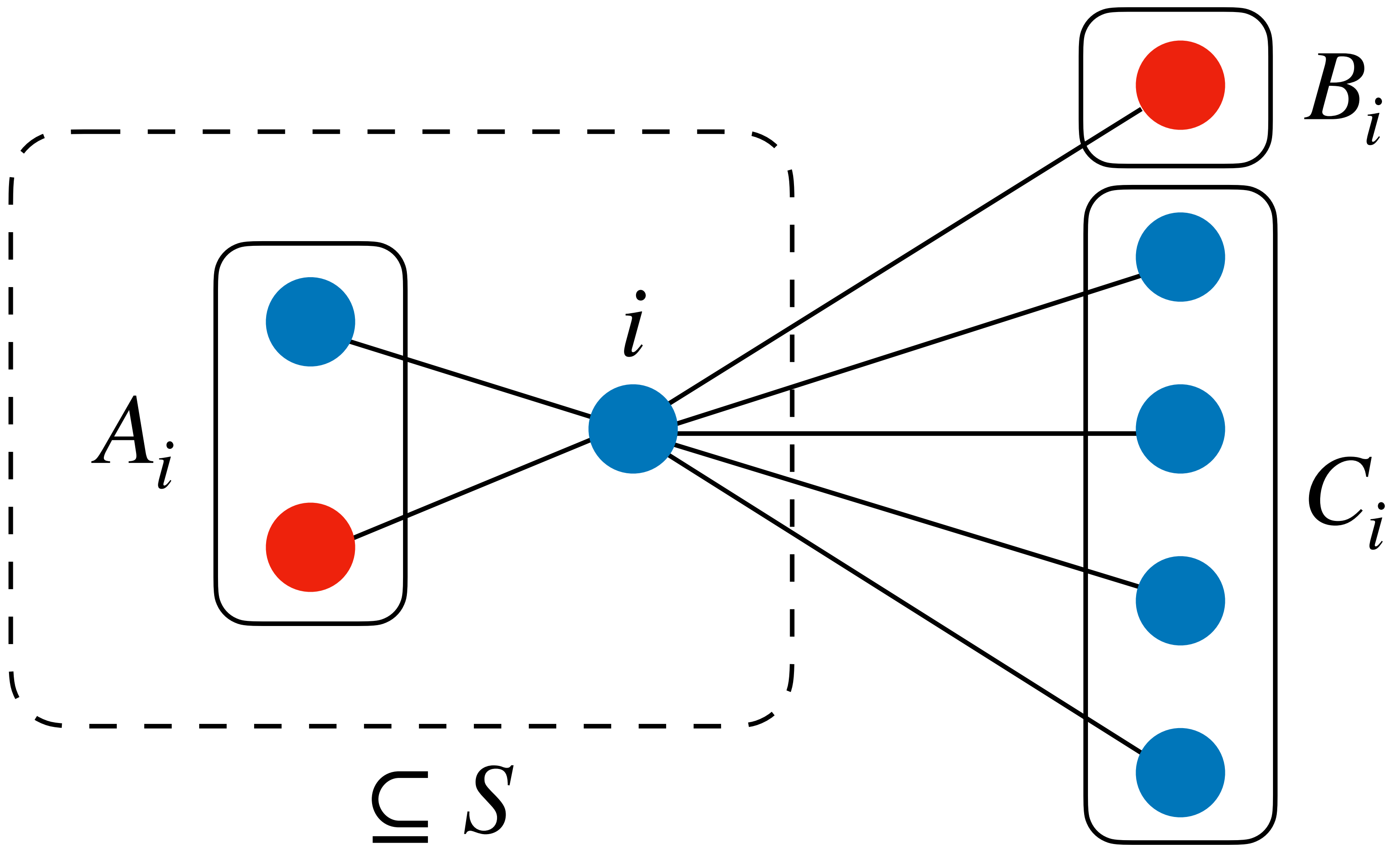

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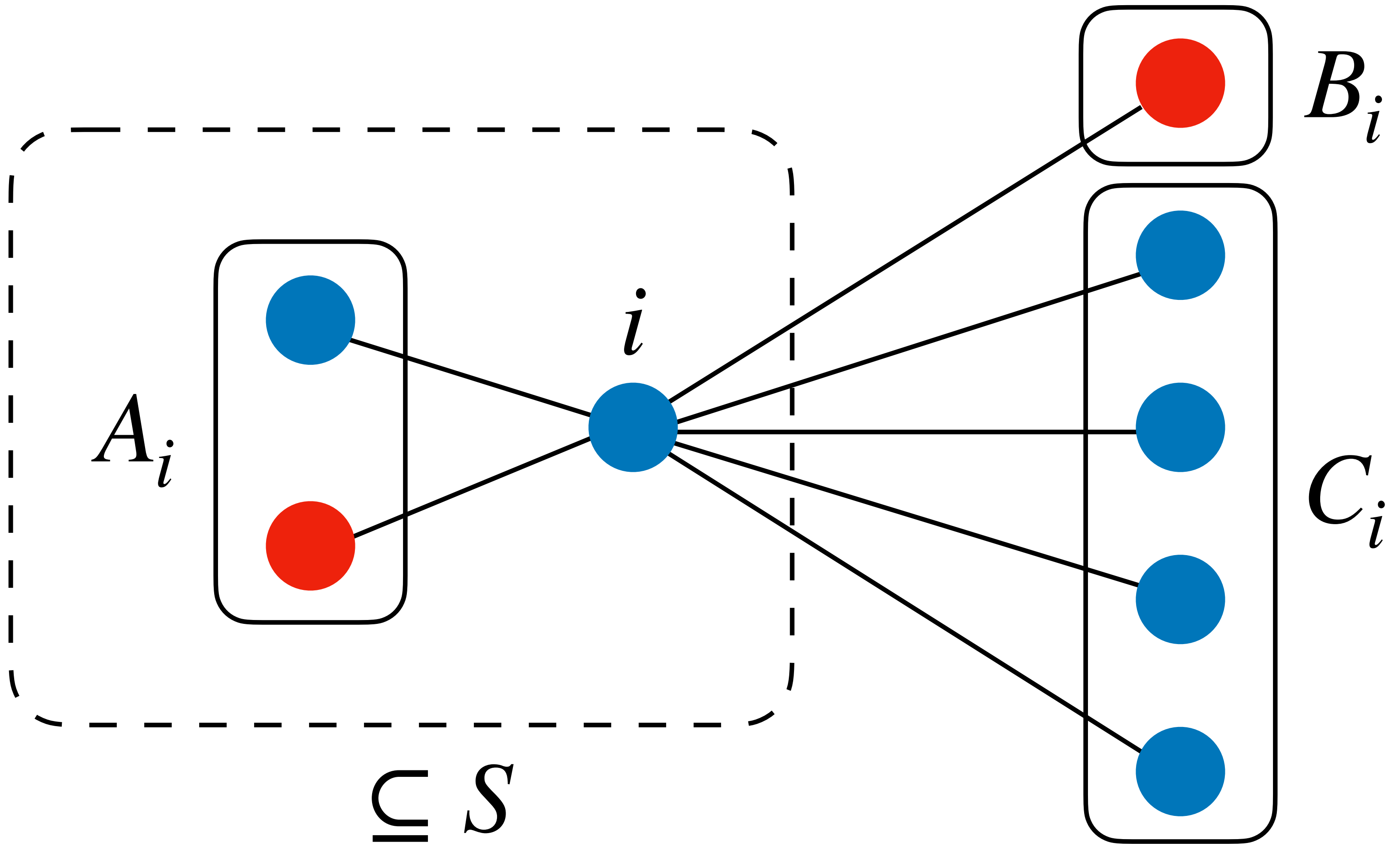

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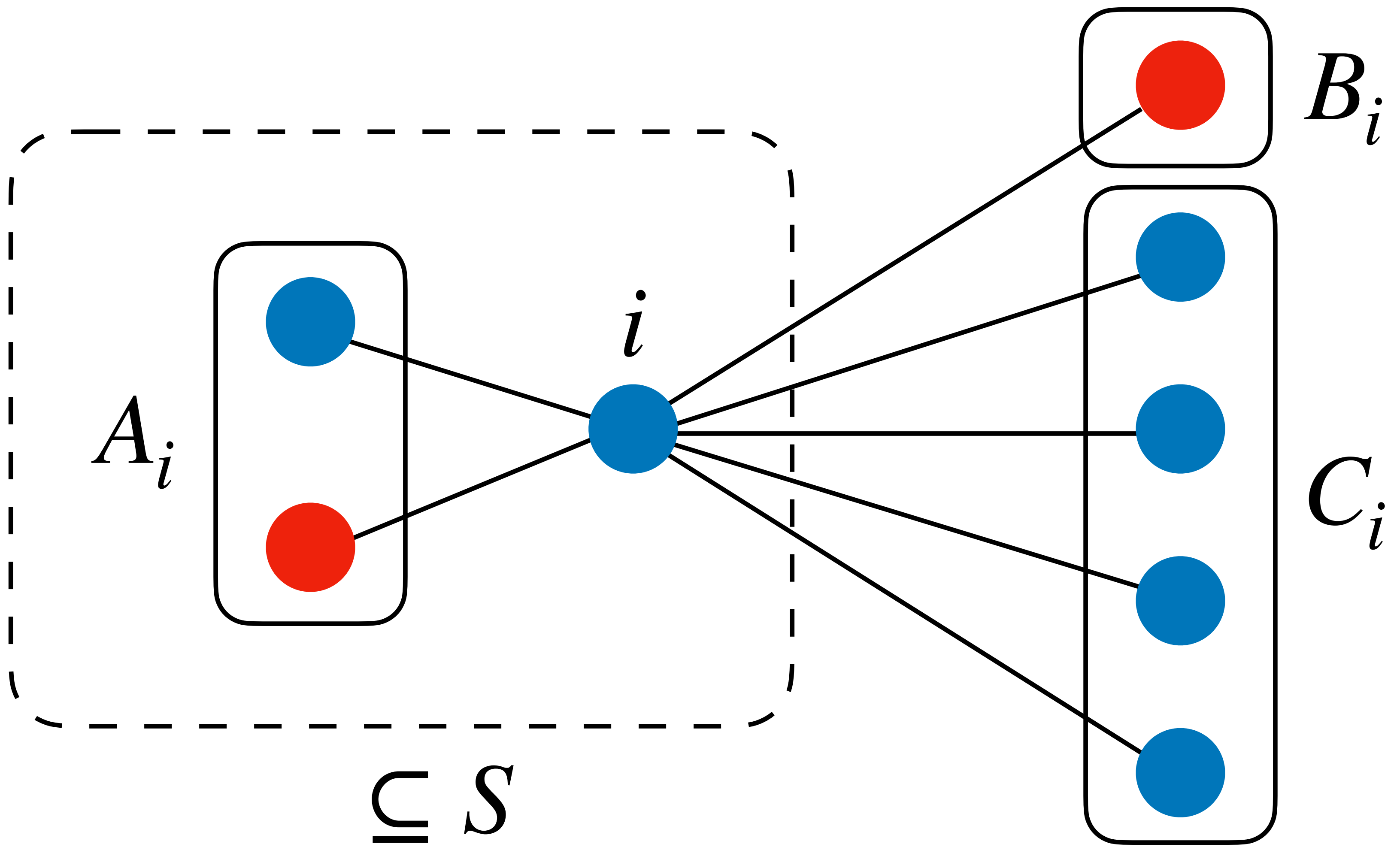

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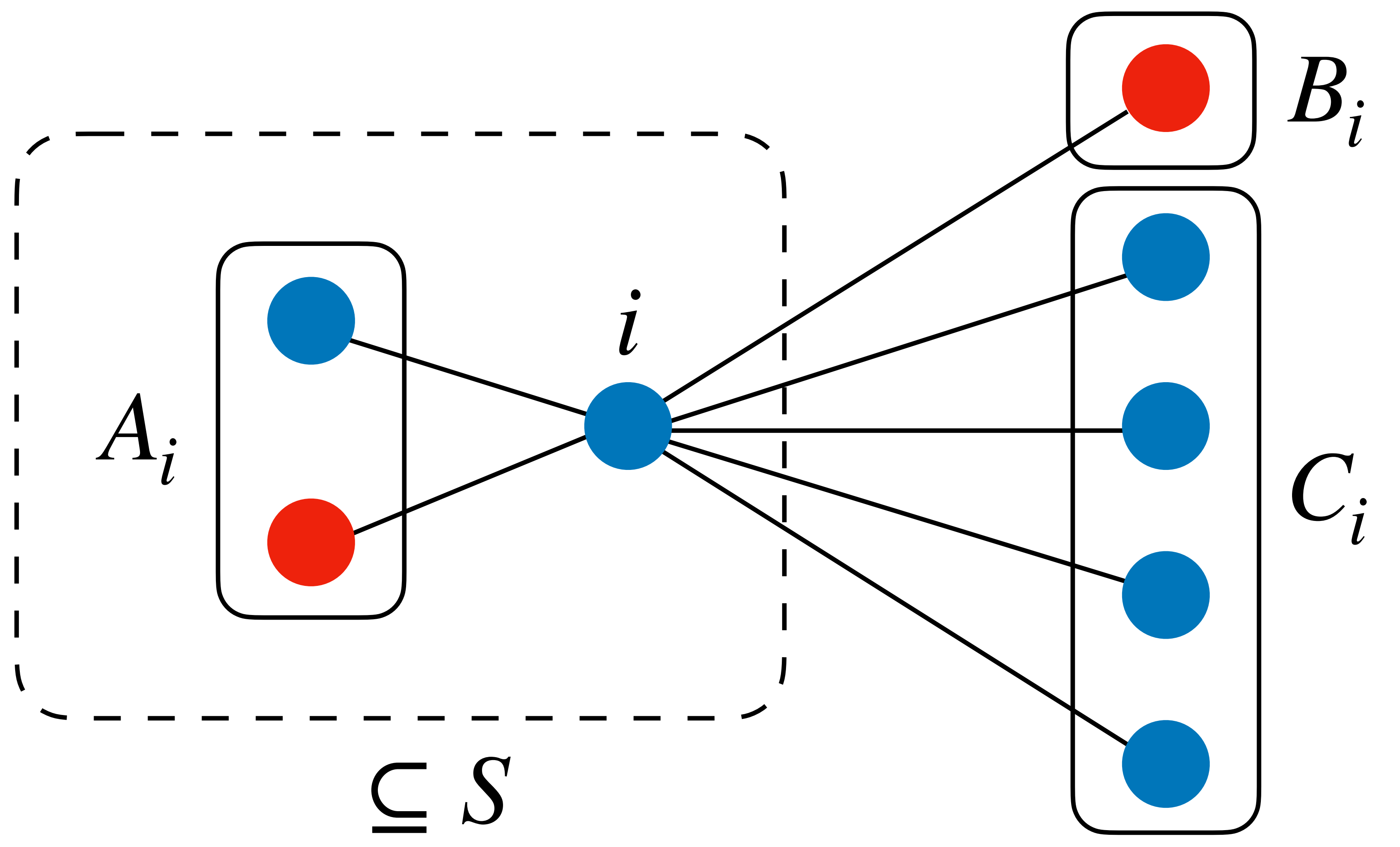

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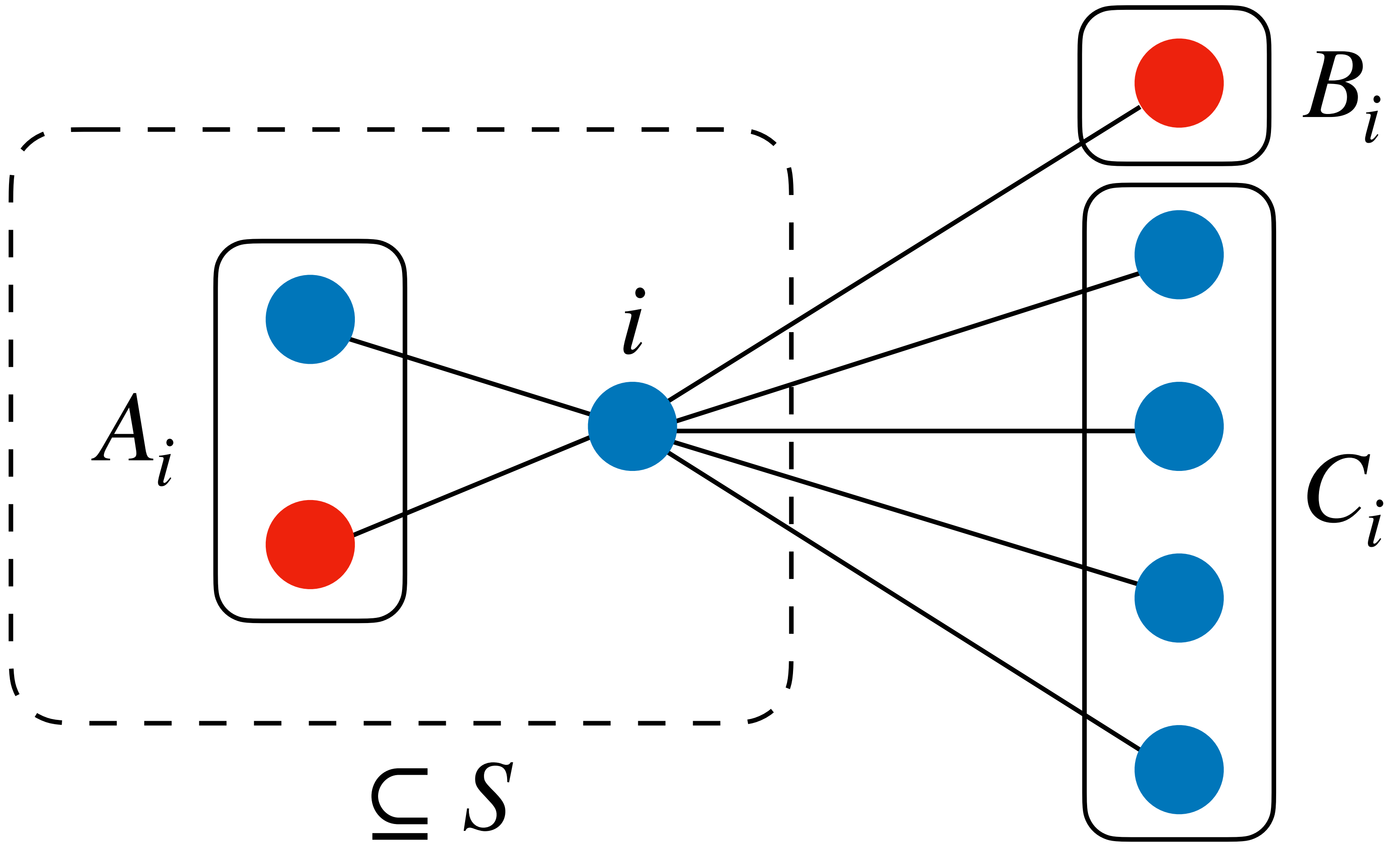
So local gain $\Delta_i \geq (2Z - d)_+$.



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Note: triangle inequality also crucial in [FKL'02, Florén'16].

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